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Improvement of a prior-knowledge-based method for reconstructing non-compactly support functions

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Abstract—**For improving the prior discrete Fourier transform (PDFT) algorithm in reconstructing small-scale portions of interest in a compactly-supported function from limited Fourier-transform data, we had successfully developed a promising technique based on a linear combination of differently-weighted spatial frequency components of the reconstructed image. This approach has been proved and demonstrated a great potential in resolution enhancement, while in this paper we further extend it for being applicable to the imaging problem with non-compactly-supported object functions. The success of this extension is essentially driven by the fact that the support domain of object functions fully occupies the image scope to which the measured data corresponds. Our newly-developed technique is tested on one- and two-dimensional simulations.**

I. INTRODUCTION

There have been many image reconstruction techniques proposed and implemented in applications such as medical imaging, remote sensing, nondestructive testing, and radar. Among all successful imaging techniques, one typical concentration is to overcome the ambiguity of image reconstruction from limited Fourier data. For this, we had developed a promising image reconstruction model for combating a challenging problem of imaging small-scaled portions in a compactly-supported function, based on a linear combination of differently-weighted spatial frequency components of the reconstructed image [1]. This new approach in principle extends the prior discrete Fourier transform (PDFT) algorithm [2]–[9] by allowing different weight functions to modulate the different spatial frequency components of the reconstructed image.

For the purpose of explanation and better understanding, let $f(x)$ be an object function supported on the interval $[-\pi, \pi]$ of the real line, and the limited number of Fourier data values be

$$F(k_n) = \int_{-\pi}^{\pi} f(x) \exp(-jxk_n) dx \,, \tag{1}$$

for $n = 1, 2, \ldots, N$. In the paper [1], for improving the imaging degradation problem in reconstructing small-scale portions of interest in a compactly-supported function, the proposed linear reconstruction estimator takes the form of

$$\hat{f}(x) = \sum_{n=1}^{N} a_n b_n(x) \,, \tag{2}$$

where the weighted spatial frequency components $b_n(x)$ are represented by

$$b_n(x) = w_n(x) \exp(jxk_n) \tag{3}$$

and the coefficients a_n satisfy the equations

$$F(k_m) = \sum_{n=1}^{N} a_n W_n(k_m - k_n) \,. \tag{4}$$

$w_n(x)$ are selected non-negative window functions, and their Fourier transforms are $W_n(k)$. As compared to the PDFT algorithm having only a uniform weight function,

$$\hat{f}_{\mathrm{PDFT}}(x) = w(x) \sum_{n=1}^{N} a_n \exp(jxk_n) \,, \tag{5}$$

the imaging algorithm in Eqn. (2) can typically give a superior image resolution. The resolution enhancement of this approach is driven by a specific mechanism of allowing each spatial frequency component to more suitably contribute to its own region in the image reconstruction procedure.

The image reconstruction technique as mentioned above has been proved and demonstrated a great potential in reconstructing small-scaled portions in a compactly-supported function from limited Fourier-transform values. However, it is typically the case that the object function's support domain may fully occupy the image scope to which the measured data corresponds. For realistic issues, the recovery of a high-resolution image for small-scaled portions in a non-compactly-supported object function becomes demanded and even more challenging. In particular for resolving this difficulty, in this paper we further extend our previously-developed imaging algorithm for being applicable to the imaging problem with non-compactly-supported object functions. The incorporation of profile-oriented prior knowledge into each complex exponential basis, for better accommodating its own reconstruction domain, is still the core principle of the newly-developed image reconstruction technique, but requiring more complex strategy in choosing the prior weights. From one- and two-dimensional simulations, our newly-developed technique clearly presented its success in resolution enhancement.

The paper is organized as follows. Section 2 introduces the theoretical and implementational background behind our proposed technique using the notation of the one-dimensional

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problem. In Section 3 we demonstrate its successful tests by one-dimensional and two-dimensional simulations.

II. THEORETICAL AND IMPLEMENTATIONAL BACKGROUND

If we consider an object function $f(x)$ having a finite support Ω, in which the small region, denoted S, containing fine-scale features of interest is enclosed by a large smoother background, denoted L. Suppose also that the set $\{k_n\}$ of spatial frequencies can be divided into two sets: one with lower values of k_n, denoted as LF, and the other one with higher values of k_n, denoted as HF. Then, according to our research findings proposed in the paper [1], Eqn. (2) can be written in the form of

$$\begin{aligned} \hat{f}(x) &= \chi_L(x) \sum_{n|k_n \in LF} a_n \exp(jxk_n) \\ &+ \chi_S(x) \sum_{n|k_n \in HF} a_n \exp(jxk_n) \end{aligned} \tag{6}$$

or

$$\begin{aligned} \hat{f}(x) &= \chi_L(x) \sum_{n|k_n \in LF} a_n \exp(jxk_n) \\ &+ \chi_S(x) \sum_{n|k_n \in LF+HF} a_n \exp(jxk_n)\,. \end{aligned} \tag{7}$$

Substituting Eqn. (6) into Eqn. (4) can give

$$\begin{aligned} F(k_m) = &\sum_{n|k_n \in LF} a_n \int_{\chi_L} \exp(-jx(k_m - k_n))dx \\ &+ \sum_{n|k_n \in HF} a_n \int_{\chi_S} \exp(-jx(k_m - k_n))dx\,. \end{aligned} \tag{8}$$

On the other hand, the substitution of Eqn. (7) into Eqn. (4) will have

$$\begin{aligned} F(k_m) = &\sum_{n|k_n \in LF} a_n \int_{\chi_L} \exp(-jx(k_m - k_n))dx \\ &+ \sum_{n|k_n \in LF+HF} a_n \int_{\chi_S} \exp(-jx(k_m - k_n))dx\,. \end{aligned} \tag{9}$$

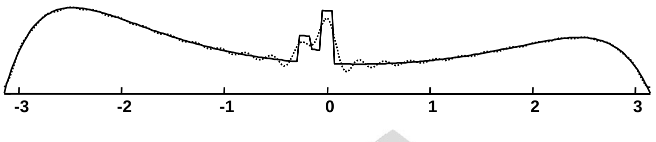

Fig. 1. The object function (solid line) and its DFT estimate (dotted line).

To obtain a high-resolution image by Eqn. (6) or Eqn. (7), the choice of weight functions $\chi_L(x)$ and $\chi_S(x)$ typically requires that $\chi_L \cap \chi_S = \emptyset$ and $\chi_L + \chi_S \approx \chi_\Omega$. If considering the image reconstruction of a compactly-supported function $f(x)$ with $\chi_\Omega(x) = [-\nu, \nu]$, it can be difficult to acquire the true support domain of $f(x)$ somehow, such as wide but low-amplitude smooth background. However, on the other hand, reconstructing a non-compactly-supported function $f(x)$ will simply take $\chi_L(x) + \chi_S(x) = [-\pi, \pi]$. With an appropriate selection of $\chi_S(x)$, in principle accommodating small-scaled portions accurately and completely, Eqn. (6) or Eqn. (7) can typically give an good result. If the profile-features of $f(x)$ over the region $\chi_S(x)$ include non-negligible low-spatial frequency components, Eqn. (7) is more recommended for reducing the artificial effectiveness of image estimation over $\chi_S(x)$, alternatively in the form of

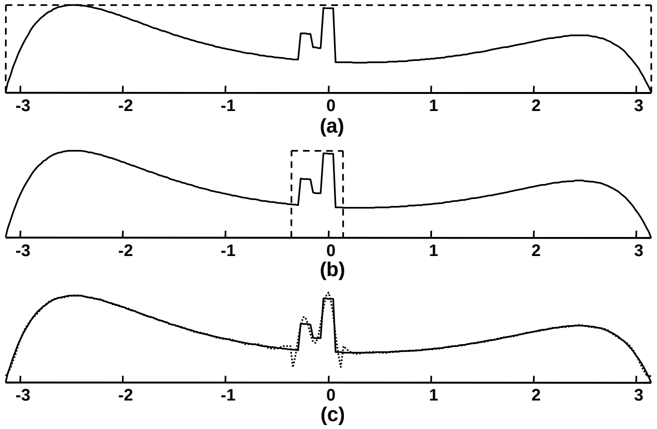

Fig. 2. An example of reconstructing a one-dimensional object function (solid line) by our proposed method, (a) the flat window function χ_L (dashed line), (b) the flat window function χ_S (dashed line), and (c) the estimate by our proposed method (dotted line) with χ_L in (a) for complex exponentials of frequencies $\{-24, \ldots, 0, \ldots, 24\}$ and χ_S in (b) for otherwise.

$$\begin{aligned} \hat{f}(x) &= \mathrm{DFT}(a_n|\, k_n \in LF) \\ &+ \chi_S(x) \sum_{n|k_n \in HF} a_n \exp(jxk_n)\,, \end{aligned} \tag{10}$$

by allowing the contribution of the low spatial frequency components over the region χ_S for the reconstructed image. DFT indicates the discrete Fourier transform.

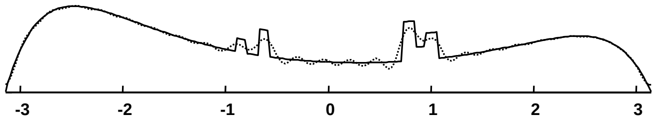

Fig. 3. The object function (solid line) and its DFT estimate (dotted line).

III. ONE- AND TWO-DIMENSIONAL SIMULATIONS

To test our findings under more realistic condition, we applied our newly-developed technique to computed projection data. The investigation was acquired for test purposes only, while it virtually presents its practical applicability. For the one-dimensional simulations, let's assume that 51 low-pass Fourier-transform data of frequencies, $\{-25, \ldots, 0, \ldots, 25\}$, are used in the image reconstruction. The typical test case is that the object is characterized as a small and finer part embedded in a large non-compactly smooth background. For the example having one fine-scale portion embedded in the background, as shown in Fig. 1 and Fig. 2, and the other one

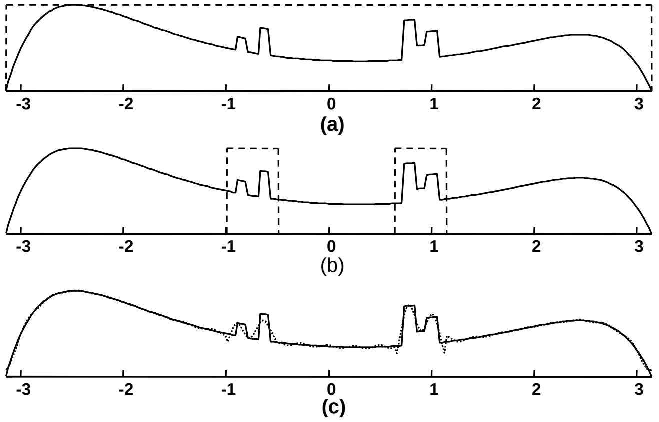

Fig. 4. An example of reconstructing a one-dimensional object function (solid line) by our proposed method, (a) the flat window function χ_L (dashed line), (b) the flat window function χ_S (dashed line), and (c) the estimate by our proposed method (dotted line) with χ_L in (a) for complex exponentials of frequencies $\{-23,\ldots,0,\ldots,23\}$ and χ_S in (b) for otherwise.

having two fine-scale portions embedded in the background, as shown in Fig. 3 and Fig. 4, both the image reconstructions by our proposed technique essentially give superior resolution enhancement than that by the DFT. From the quantitative point of view of the difference between the original and reconstructed images, the root mean square error (RMSE) can be typically used to determine the goodness of image reconstruction. In our first one-dimensional simulations, the RMSEs of DFT estimate in Fig. 1 and the result by our proposed method in Fig. 2 are 0.043 and 0.033 respectively. On the other hand, for our second one-dimensional simulations, the RMSEs of DFT estimate in Fig. 3 and the result by our proposed method in Fig. 4 are 0.044 and 0.035 respectively. Moreover, the two-dimensional simulations, as shown in Fig. 5 and Fig. 6, also presents a great potential of our proposed technique. The RMSEs of the two-dimensional DFT estimate in Fig. 5 and the result by our proposed method in Fig. 6 are 13.703 and 0.012 respectively.

IV. CONCLUSIONS

Based on the previously-developed image reconstruction of resolving small-scale features within a portion of a compactly-supported function from limited Fourier-transform values, we have successfully extended it into a new approach for accommodating those applications of which the object function can be non-compactly-supported. Comparisons are made with the DFT image reconstruction, and clearly show its great potential in resolution enhancement. For applications in which non-negligible low spatial frequency components exist within the small portions of interest, the prior window associated with low spatial frequency components is recommended as the one having its support as the object's true domain for an acceptable image resolution.

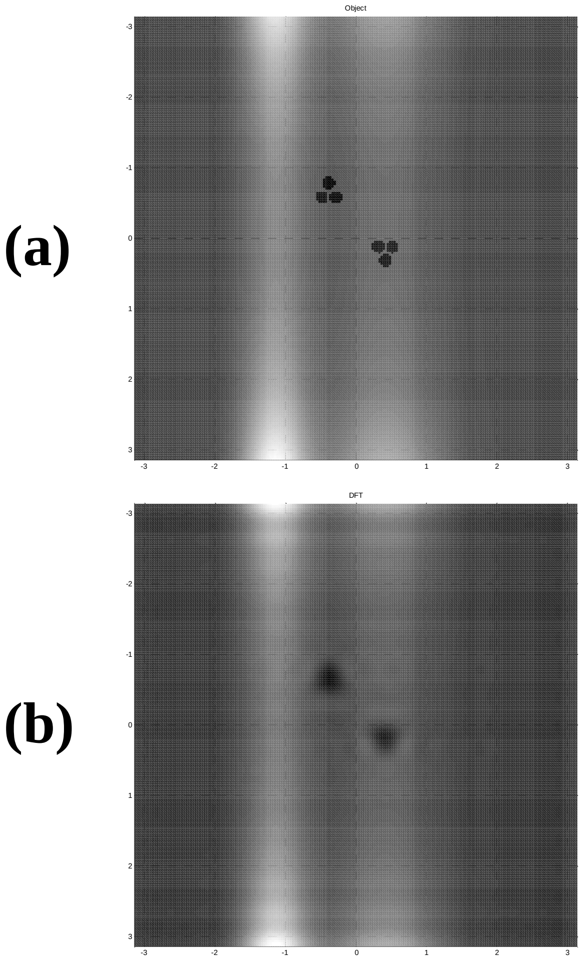

Fig. 5. An example of reconstructing a two-dimensional object function by the DFT, (a) the object function, (b) the DFT estimate.

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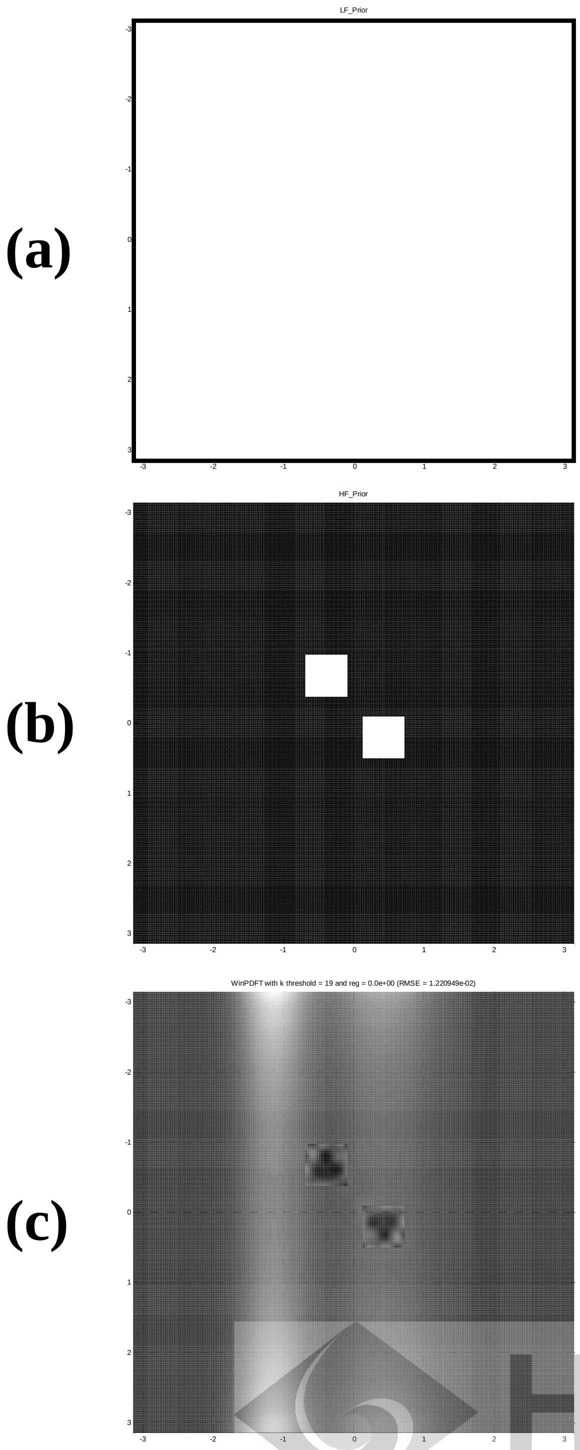

Fig. 6. An example of reconstructing a two-dimensional object function by our proposed method, (a) the flat window function χ_L, (b) the flat window function χ_S, and (c) the estimate by our proposed method.