Lengths and Alphabet Size of Recursively Constructed Complex Golay Sequences

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***Abstract* —Golay complementary sequences have been used in synchronization, channel estimation, and peak-to-average power ratio reduction for multicarrier signals. The length and alphabet size of recursively constructed complex Golay complementary sequences are determined by the input Golay pairs used in the construction. The length of the constructed Golay sequence will contain product of the lengths of the input Golay pairs as its factor. The alphabet size will grow when the number of QAM input Golay pairs increases. The growth of the length and alphabet size are analyzed. An alphabet size reduction method is proposed.[1]**

I. INTRODUCTION

A Golay sequence pair (*A*, *B*) of length *N* has the property that the sum of the aperiodic autocorrelation functions of two pairing sequences is an impulse function:

$$R_{\boldsymbol{A}}(u) + R_{\boldsymbol{B}}(u) = 0, \quad u \neq 0.$$

$$R_{\boldsymbol{A}}(u) = \sum_{i=0}^{N\text{-}u\text{-}1} A_i A_{i+u}^{*} \quad 0 \le u \le n-1, \;\; R_{\boldsymbol{A}}(-u) = R_{\boldsymbol{A}}^{*}(u). \quad (1)$$

Golay sequences [1][2] have many applications in communications systems due to their aperiodic autocorrelation function property. Golay sequences have been employed in IEEE 802.16e for uplink sounding in the frequency domain to create low peak power time waveform, and have been used in pairs in IEEE 802.15.3c for synchronization and channel estimation.

One of the central theoretical questions in Golay sequence research, raised by Gibson and Jedwab [2], is:
"*For which lengths and over which alphabets does a Golay sequence pair exist?*"

Although Golay sequences have been known to exist with PSK and QAM alphabets for various lengths, and a particularly rich amount of knowledge is known for lengths 2^m, over all PSK sequences, the answer to the question above is far from being complete. The lack of understanding leads to degraded performance in applications [3]. The uplink sounding sequences in 802.16e are punctured subsequences of a length 2048 BPSK Golay sequences, which are no longer Golay sequences and the corresponding time waveforms cannot maintain the 3dB peak-to-average power ratio (PAPR) upper bound. They can actually be replaced by recursively constructed QPSK complex Golay sequences with inserted zeros that maintain the complementary property and the 3dB PAPR upper bound [3].

To better understand Golay sequences and prevent the performance degradation in practical applications similar to the case in 802.16e [3], we study the lengths and alphabet size of recursively constructed complex Golay sequences obtained from BPSK, QPSK, and 16-QAM Golay pairs.

II. THE RECURSIVE CONSTRUCTION

Given two Golay pairs (*A*, *B*), (*C*, *D*) of any lengths and alphabets, a Golay pair (*U*, *V*) can be constructed recursively [4][5]:

$$\boldsymbol{U}(z) = \boldsymbol{A}(z^{\alpha})\boldsymbol{C}(z^{\beta}) + \boldsymbol{B}(z^{\alpha})\boldsymbol{D}(z^{\beta})z^{\gamma},$$
$$\boldsymbol{V}(z) = \boldsymbol{A}(z^{\alpha})\bar{\boldsymbol{D}}^{*}(z^{\beta}) - \boldsymbol{B}(z^{\alpha})\bar{\boldsymbol{C}}^{*}(z^{\beta})z^{\gamma}.$$
$$\text{where } \boldsymbol{A}(z) = A_0 + A_1 z + A_2 z^2 + \ldots + A_{N-1}z^{N-1}. \quad (2)$$

The parameters (α, β, γ) can be arbitrary integers.

If the input pairs (*A*, *B*), (*C*, *D*) have lengths *N*, *M*, the output pair (*U*, *V*) can have length 2NM for block concatenation, block interleaving, and element interleaving constructions without "symbol collision". The parameters for block concatenation/ block interleaving/ element interleaving are $(\alpha, \beta, \gamma) = (1, N, NM), (1, 2N, N), (2, 2N, 1)$. These constructions create output sequences whose elements are products of symbols (or their conjugates) from two input sequences. For example, the block concatenation construction is as below [4][5].

$$U = [AC_0, AC_1, \ldots, AC_{M-1}, BD_0, BD_1, \ldots BD_{M-1}]$$
$$V = [AD_{M-1}^{*}, AD_{M-2}^{*}, \ldots, AD_0^{*}, -BC_{M-1}^{*}, -BC_{M\text{-}2}^{*}, \ldots, -BC_0^{*}] \quad (3)$$

where $A = [A_0, A_1, \ldots A_{N-1}]$, $AC_0 = [A_0C_0, A_1C_0, \ldots A_{N-1}C_0]$, and $D_k^{*} = conj(D_k)$ is the complex conjugate of D_k.

Consider a Golay pair (U^m, V^m) obtained from *m* iterations of recursive constructions, see Fig. 1. In the *k*th iteration, $(U^{k-1}, V^{k-1}), (A^k, B^k)$ are the input Golay pairs, (U^k, V^k) is the output pair. Let (A^k, B^k), $0 \le k \le m$ be BPSK, QPSK, or 16-QAM Golay pairs. Let

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$(U^0, V^0) = (A^0, B^0)$, and let p be the number of 16-QAM Golay pairs among (A^k, B^k), $0 \le k \le m$. The alphabet size of the output pair (U^m, V^m) is defined as the number of symbols in the smallest square QAM constellation that contains all possible symbols of (U^m, V^m).

Since each symbol in the output Golay sequences U^m, V^m will be the product of m+1 symbols (some may be conjugated) from the input Golay sequences A^k, B^k, $0 \le k \le m$. When at least two input Golay pairs are 16-QAM Golay pairs ($p \ge 2$), the alphabet of the output Golay sequences will be bigger than 16-QAM, but can be scaled/rotated so that all symbols fall on the standard positions of some QAM constellation, where the real part and imaginary part are both odd-valued. The size of the QAM constellation is analyzed as a function of p. To minimize the alphabet size, 16-QAM Golay sequence pairs of lengths 7 and 9 without using the four corner points in the 16-QAM constellation ($\pm 3 \pm 3j$) as described in [5][6] are employed as input Golay pairs with "reduced16-QAM" symbols.

Let the lengths of input Golay sequences A^k, B^k be L_k. When the recursive construction in all iterations are either block concatenation, block interleaving, or element interleaving, the length of the output Golay sequences U^m, V^m will be $2^m \prod_{k=0}^{m} L_k$. Both the sequence length and alphabet symbol of the output Golay pair contain the product of the lengths and alphabet symbols of input Golay pairs.

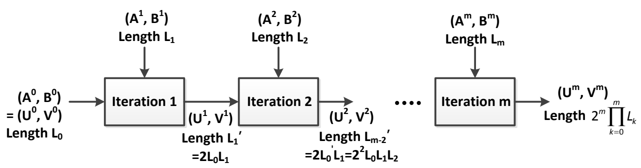

Fig. 1 m-stage recursive construction
Input pairs $(A^0, B^0),\ldots,(A^m, B^m)$ Output pair (U^m, V^m)

III. The Alphabet Size and Sequence Lengths

The scaled products of p 16-QAM symbols and reduced 16-QAM symbols are shown in Fig. 2.

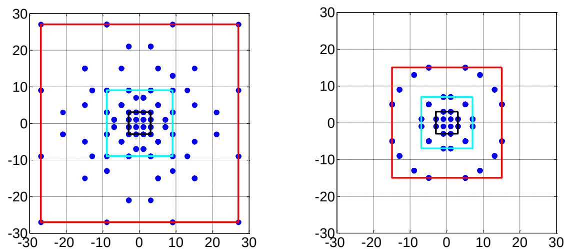

Fig. 2 Scaled Products of p=1, 2, 3 (a) 16-QAM symbols (b) Reduced 16-QAM symbols

The size of the square QAM constellation (the alphabet size) may be restricted to 4^q if the constructed sequences U^m, V^m are intended for data transmission, where each symbol is associated with a binary bit string; the size may be $4K^2, K \in Z$ if the sequences are used for other special purposes (e. g. channel estimation), but not for data transmission. In general $4K^2 \le 4^q$. Let q be the parameter obtained by taking base four log value of the alphabet size. There will be two parameter values, $q_1 = \log_4 4^q, q_2 = \log_4 4K^2, q_1 \ge q_2$.

To reduce the alphabet size of (U^m, V^m), note that among all known 16-QAM Golay seed pairs at lengths 7, 9, 15 where no BPSK or QPSK Golay pair exists, 16-QAM Golay pairs exist at each length containing no symbols at the four "corners" of the 16-QAM constellation, $\{\pm 3 \pm 3j\}$ [5][6]. In other words, they employ a "reduced 16-QAM" alphabet that contains 12 out of the 16-QAM symbols. Seed pairs are Golay pairs that cannot be viewed as being obtained from recursive construction using shorter Golay pairs. When only the reduced 16-QAM Golay seed pairs are used as the p 16-QAM pairs in the m stage recursive constructions, the alphabet size of the output pair (U^m, V^m) is also reduced. The alphabet size and size differences (in number of QAM symbols) as a function of p (number of input 16-QAM Golay pairs) is given in Table 1. The difference increases rapidly as p increases. The alphabet size reduction obtained from the use of reduced 16-QAM Golay sequences as input Golay sequences is quite significant.

Table 1: Compare Alphabet Size of Output Golay Pair when using 16-QAM or Reduced 16-QAM Input Golay Pairs

p	Input 16-QAM Golay Pairs Output $4K^2$-QAM	Input Reduced-16-QAM Golay Pairs, Output $4K^2$-QAM	Difference (QAM symbols)
2	100-QAM	64-QAM	36-QAM(36%)
3	784-QAM	256-QAM	528-QAM(67.35%)
4	6724-QAM	1296-QAM	5428-QAM(80.73%)
5	59536-QAM	6400-QAM	53136-QAM(89.25%)
6	532900-QAM	30976-QAM	501924-QAM(94.19%)
7	4787344-QAM	156816-QAM	4630528-QAM(96.7%)

The alphabet size of the output Golay pair when all input 16-QAM Golay pairs employ reduced 16-QAM alphabets can also be expressed either as $4^{\tilde{q}}$ or $4\tilde{K}^2$. Two more q parameter values can be defined for the "reduced QAM constellation" as: $q_3 = \log_4 4^{\tilde{q}}, q_4 = \log_4 4\tilde{K}^2, q_3 \ge q_4, q_3 \le q_1$, $q_4 \le q_2$. The computation of the q parameters as a function of the number of the 16-QAM input Golay pairs p is carried out for $p \ge 2$. A maximum magnitude symbol can take on the value $(3+3j)^p/(1+j)^{p-1}$ or $(1+3j)^p/(1+j)^{p-1}$ for the reduced QAM constellation. The results are as follows:

$$q_1 = \log_4(4^q) = ceiling(\log_2(3^p + 1)).$$
$$q_2 = \log_4(4K^2) = \log_2(3^p + 1).$$
$$q_3 = \log_4(4^{\tilde{q}}) = ceiling(log_2(M(p) + 1)).$$
$$q_4 = \log_4(4\tilde{K}^2) = log_2(M(p) + 1).$$
$$\text{where } M(p) = \max(|\mathrm{Re}\,al(S)|, |\mathrm{Im}\,ag(S)|). \qquad (4)$$
$$S = (1 + 3j)^p / (1 + j)^{p-1}.$$
$$q_5 = \log_4(4\tilde{K}^2 - \text{No. of points out of radius } M(p) \text{ circle}).$$

Fig. 3 plots all q parameters as functions of p.

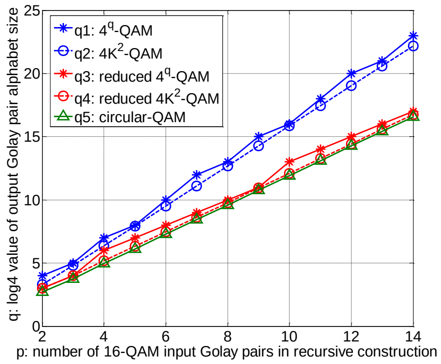

Fig. 3 Alphabet size of QAM Golay sequences recursively constructed from p 16-QAM or reduced 16-QAM input Golay pairs and an arbitrary number of BPSK, QPSK Golay pairs.

Table 2 gives the possible lengths of Golay sequences constructed from BPSK, QPSK, and 16-QAM seed pairs. The lengths of BPSK Golay sequences can be $2^{K_0}10^{K_1}26^{K_2}$ with arbitrary integers K_0, K_1, K_2, but the lengths of QPSK and QAM Golay sequences lengths will require a sufficiently large K_0 value as determined computationally through a program "gseq" [7]. The requirement of a large K_0 value (length factors contain large powers of two) is a result of the two-pair recursive construction, where two Golay pairs of lengths N and M are combined to form a Golay pair with length $2NM$. The lengths of Golay sequences obtained from repeated applications of this construction will naturally contain large powers of two factors. For example, although there exist length 3 QPSK Golay sequences, length 9 QPSK Golay sequences cannot be constructed from two QPSK Golay pairs of length 3, since the constructed QPSK Golay pair would have length 2(3)(3)=18. However, length 9 16-QAM Golay sequences have been found [6]. New seed pairs discovered in the future will further increase the possible lengths of Golay sequences.

Table 2: Lengths of recursively constructed Golay sequences from seed pairs

Alphabet	Seed Pair Lengths	Lengths of Golay sequences constructed from seed pairs
BPSK	1,10,26	$2^{K_0}10^{K_1}26^{K_2}$
QPSK	3,5,11,13	$2^{K_0}10^{K_1}26^{K_2}3^{K_3}5^{K_4}11^{K_5}13^{K_6}$
≥16-QAM	7,9,15	$2^{K_0}10^{K_1}26^{K_2}3^{K_3}5^{K_4}11^{K_5}13^{K_6}7^{K_7}9^{K_8}15^{K_9}$

Fig. 4 contains the lengths and alphabet size (take log value with base four) of recursively constructed complex Golay sequences up to length 168. The use of larger alphabets leads to more sequence lengths. For example, all BPSK Golay sequences must have even lengths, whereas QPSK Golay sequences can take on odd lengths such as 3, 5, 11, and 13. Although BPSK symbols can be viewed as special case QPSK symbols, in Table 2 and Fig. 4 the "QPSK sequences" are those that are not BPSK sequences.

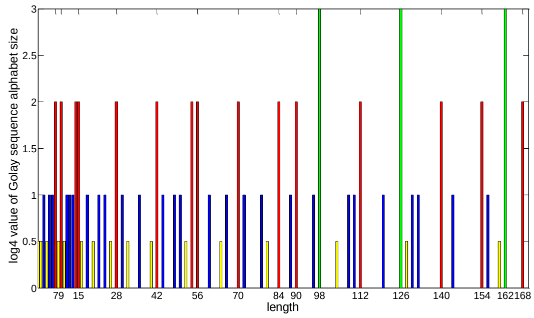

Fig. 4 Lengths and alphabet size of recursively constructed complex Golay sequences. The alphabet size is the log 4 value of the number of points in the square QAM constellations. The alphabet size parameters for BPSK, QPSK, 16-QAM and 64-QAM alphabets are 0.5, 1, 2, 3, respectively.

IV. CONCLUSION

The alphabet size and sequence lengths of recursively constructed complex Golay sequences are analyzed. The use of multiple QAM input Golay pairs in the recursive construction allows the output Golay pairs to reach more new lengths at the cost of alphabet size expansion. The use of reduced magnitude input QAM Golay pairs can significantly reduce the alphabet size of the output Golay sequences. A program "gseq" is created to construct Golay sequences for a given length with the smallest alphabet.

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