An Efficient Least-Squares Design of Two-Channel Quadrature Mirror Filters Using IIR All-Pass Filters

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***Abstract* —The design of two-channel quadrature mirror filter banks can be constructed by using IIR all-pass filter in the least-squares sense without causing magnitude distortions. The complex nonlinear phase optimization problem is first converted into solve a system of linear equations. Then, the associated matrices involved in the set of linear equations can be further formulated as a Toeplitz-plus-Hankel form such that a matrix inversion is avoided. Simulation results confirm that the proposed method achieves good performance as well as with effectiveness.**[1]

I. INTRODUCTION

Quadrature mirror filter (QMF) banks have been received much attention due to the effectiveness and applications in multirate signal processing [1]-[3] over the past two decades. The designs of two-channel QMF banks using conventional FIR or IIR structures for the prototype low-pass analysis filter usually result both magnitude and phase distortions. Recently in the literatures [4]-[7], the two-channel perfect reconstruction QMF banks can be successfully constructed using IIR all-pass digital filters.

It has been shown in [4][6][7] that the parallel combination of two IIR all-pass filters can be developed to simultaneously meet both the magnitude and phase specifications without causing distortions. The main advantage based on IIR all-pass lies in the design problem can be focused on the phase approximation without resulting magnitude distortion [6][7]. In [7], the design problem is approximately formulated as to find the real coefficients of the IIR digital all-pass filter from a linear subspace related phase objective function. Using a variant of Karmarkar's algorithm, the optimization problem can be solved efficiently.

There are several approaches proposed for the design of IIR all-pass filter [6]-[10]. In this work, the QMF banks optimization based on IIR all-pass filter is first simplified as a linear phase optimization. Then the optimal IIR all-pass filter coefficients can be formulated by solving a system of linear equations comprising a Toeplitz-plus-Hankel matrix. Consequently, the solution of such a system of linear equations can be solved using Cholesky or Levinson decomposition technique [12], [13] that requires only $O(N^2)$ complexity [11]. It is computationally efficient as compared to solve the system of linear equations by directly computing a matrix inversion which involves $O(N^3)$ complexity. On the other hand, matrix inversion computation may cause numerical problems when the filter order is large. Therefore, the method proposed by Kidambi [8][11] is not only efficient but also robust. In this paper, we exploited the efficient method to the design of two-channel QMF bank based on IIR all-pass filters.

II. PROBLEM FORMULATION

The two-channel IIR all-pass based QMF bank is shown in Fig. 1. The analysis low-pass filter $H_0(z)$ and high-pass filter $H_1(z)$ of the QMF bank are composed of a parallel combination of two IIR all-pass filters shown below [6] [7]:

$$H_0(z)=\frac{1}{2}\left[A_0\left(z^2\right)+z^{-1}A_1\left(z^2\right)\right] \tag{1}$$

$$H_1(z)=\frac{1}{2}\left[A_0\left(z^2\right)-z^{-1}A_1\left(z^2\right)\right] \tag{2}$$

where $A_0\left(z^2\right)$ and $A_1\left(z^2\right)$ are two real IIR all-pass filters. The synthesis filters $F_0(z)$ and $F_1(z)$ should satisfy the relations $F_0(z)=H_0(z)$ and $F_1(z)=-H_1(z)$, respectively, such that the aliasing errors can be completely eliminated.

Consequently, the perfect reconstruction QMF bank becomes a linear and shift-invariant system with transfer function shown as:

$$M(z)=\frac{1}{2}z^{-1}A_0\left(z^2\right)A_1\left(z^2\right). \tag{3}$$

Therefore, the main concern lies in determining the coefficients of the IIR all-pass filters $A_0\left(z^2\right)$ and $A_1\left(z^2\right)$ such that the phase response $Arg\left\{M\left(e^{j\omega}\right)\right\}$ approximates to the desired phase response.

The frequency responses of IIR all-pass filters $A_0\left(z^2\right)$ and $A_1\left(z^2\right)$ are expressed as [6][7]

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$$A_i\left(e^{j2\omega}\right) = e^{-j2N_i\omega}\frac{\sum_{n=0}^{N_i} a_i(n)e^{j2n\omega}}{\sum_{n=0}^{N_i} a_i(n)e^{-j2n\omega}} = e^{j\theta_i(\omega)} \tag{4}$$

where $i = 0,1$. The design phase response $\theta_i(\omega), i = 0,1$, is concluded as

$$\begin{aligned}\theta_i(\omega) &= -2\,N_i\ \omega - 2\varphi_i(\omega),\\ &= -2\,N_i\ \omega + 2\tan^{-1}\left(\frac{\sum_{n=1}^{N_i} a_i(n)\sin(2\,n\,\omega)}{1+\sum_{n=1}^{N_i} a_i(n)\cos(2\,n\,\omega)}\right),\end{aligned} \tag{5}$$

where $a_i(0) = 1,\ \ i = 0,1$. Consequently, the frequency responses of the analysis filters can be easily obtained as

$$H_0\left(e^{j\omega}\right) = e^{j\left(\frac{\theta_0(\omega)+\theta_1(\omega)-\omega}{2}\right)}\cos\left(\frac{\theta_0(\omega)-\theta_1(\omega)+\omega}{2}\right), \tag{6}$$

$$H_1\left(e^{j\omega}\right) = j\,e^{j\left(\frac{\theta_0(\omega)+\theta_1(\omega)-\omega}{2}\right)}\sin\left(\frac{\theta_0(\omega)-\theta_1(\omega)+\omega}{2}\right). \tag{7}$$

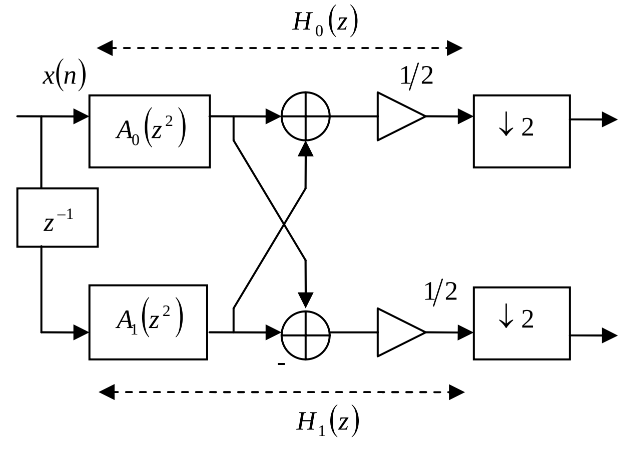
To ensure that $H_0(z)$ and $H_1(z)$ are linear-phase low-pass and high-pass filters, respectively, the phase response $\theta_i(\omega), i = 0,1$ should be chosen properly. The following conditions: $N_0 = N_1 + 1$, $\theta_i = -2N_i\ \omega \pm \omega/2$ in passband region and $\theta_i = -2N_i\ \omega \pm (\omega/2 - \pi/2)$ in stopband region which are suggested [7]. The overall frequency response of the QMF bank magnitude becomes

$$M\left(e^{j\omega}\right) = \frac{1}{2}e^{j[-\omega+\theta_0(\omega)+\theta_1(\omega)]} = \frac{1}{2}e^{-j(2N_0+2N_1+1)\omega}. \tag{8}$$

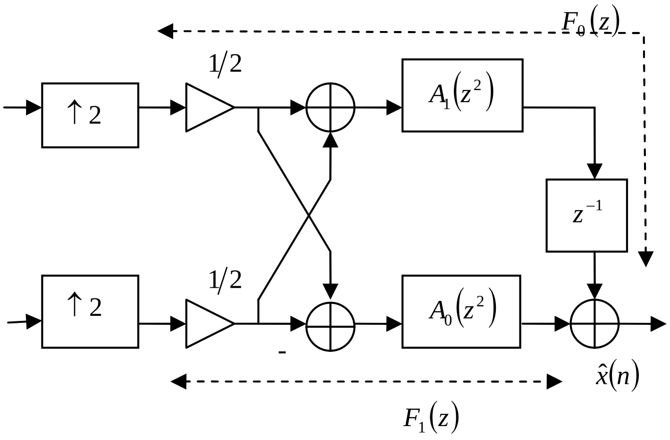
Clearly, the resulting QMF bank is a linear-phase response with group delay $g_d = 2N_0 + 2\,N_1 + 1$ and does not incur magnitude distortion.

III. QMF DESIGN USING IIR ALL-PASS FILTERS

In literature [8], an efficient iterative approach is proposed to design an IIR all-pass filter that has a least-squares or an equiripple phase error response. The method is based on formulating a weighted error reflecting the difference between the desired phase $\theta_{d,i}(\omega)$ and the phase response of the design IIR all-pass filters shown in (9). Obviously, the phase error function involves in a nonlinear optimization problem with complex procedures. Assume that the design phase $\theta_i(\omega)$ is extremely close to the desired phase $\theta_{d,i}(\omega)$, Kidambi [8] reformulated (5) as (10), where $\rho_{d,i}(\omega) = (\theta_{d,i}(\omega) + 2\,N_i\ \omega)/2, i = 0,1, \mathbf{a}_i$, $\mathbf{c}_i(\omega)$ and $\mathbf{s}_i(\omega)$



(a) analysis system



(b) synthesis system

Fig. 1 Two-channel IIR all-pass based QMF bank.

$$e(\omega) = \theta_{d,i}(\omega) + 2\,N_i\ \omega - 2\tan^{-1}\left(\frac{\sum_{n=1}^{N_i} a_i(n)\sin(2\,n\,\omega)}{1+\sum_{n=1}^{N_i} a_i(n)\cos(2\,n\,\omega)}\right). \tag{9}$$

$$\frac{\sin[\rho_{d,i}(\omega)]}{\cos[\rho_{d,i}(\omega)]} = \frac{\mathbf{a}_i^T\,\mathbf{s}_i(\omega)}{1+\mathbf{a}_i^T\mathbf{c}_i(\omega)} \tag{10}$$

are the vector representations defined below:

$$\mathbf{a}_i = [a_i(1), a_i(2), \cdots, a_i(N_i)]^T\ , \tag{11}$$

$$\mathbf{c}_i(\omega) = [\cos(2\,\omega), \cos(4\,\omega), \cdots, \cos(2N_i\ \omega)]^T, \tag{12}$$

$$\mathbf{s}_i(\omega) = [\sin(2\,\omega), \sin(4\,\omega), \cdots, \sin(2N_i\ \omega)]^T. \tag{13}$$

As a result, the objective function can be simplified as

$$E_i = \sum_{l=1}^{L_i}\left\{\mathbf{a}_i^T\ \mathbf{s}_{1,i}(\omega_l) + \sin[\rho_{d,i}(\omega_l)]\right\}^2, i = 0,1. \tag{14}$$

where $\mathbf{s}_{1,i}(\omega_l) = \sin\lfloor\rho_{d,i}(\omega_l)\rfloor \mathbf{c}_i(\omega_l) - \cos\lfloor\rho_{d,i}(\omega_l)\rfloor \mathbf{s}_i(\omega_l)$, and L_i is the frequency sampling points. By differentiating the objective function with filter coefficients, $\partial E_i / \partial a_i(n) = 0$, for $n = 1, 2, \cdots, N_i$, $i = 0,1$, a system of linear equations given by $\mathbf{Q}_i \, \mathbf{a}_i = \mathbf{d}_i$, $i = 0,1$ is easily obtained, where

$$\mathbf{Q}_i = \sum_{l=1}^{L_i} \mathbf{s}_{1,i}(\omega_l) \cdot \mathbf{s}_{1,i}^T(\omega_l), \tag{15}$$

$$\mathbf{d}_i = -\sum_{l=1}^{L_i} \sin\left[\rho_{d,i}(\omega_l)\right] \mathbf{s}_{1,i}(\omega_l). \tag{16}$$

Since $\mathbf{Q}_i$ is a real, symmetric and positive-definite matrix, a unique solution is guaranteed. Kidambi [8] [11] further expanded the matrix $\mathbf{Q}_i$ into a sum of a symmetric Toeplitz matrix and a Hankel matrix shown below:

$$\begin{aligned} Q_i(u, v) &= \sum_{l=1}^{L_i} \sin\left[\rho_{d,i}(\omega_l) - 2u\,\omega_l\right] \cdot \sin\left[\rho_{d,i}(\omega_l) - 2v\,\omega_l\right] \\ &= \frac{1}{2}\left\{\sum_{l=1}^{L_i} \cos 2(u - v)\omega_l - \sum_{l=1}^{L_i} \cos\left[2(u + v)\omega_l - 2\rho_{d,i}(\omega_l)\right]\right\} \\ &= T_i(u, v) + H_i(u, v), \quad i = 0,1. \end{aligned} \tag{17}$$

Consequently, the system of linear equations can be written as $(\mathbf{T}_i + \mathbf{H}_i)\mathbf{a}_i = \mathbf{d}_i$, $i = 0,1$. An iterative and computationally efficient algorithm involves $O(N^2)$ complexity can be used to obtain the optimal filter coefficients. For a $N_i \times N_i$ Toeplitz matrix, there are only N_i different elements required to be computed, that is, $T_i(1, v)$, $1 \le v \le N_i$. On the other hand, the Hankel matrix is not only dependent on cosine function but also dependent on the desired phase response. Considering for the phase response shown in Section 2 is designed, then $\rho_{d,i}(\omega) = \pm\,\omega/4, i = 0,1$ in passband and $\rho_{d,i}(\omega) = \pm\,\omega/4 \mp \pi/4, i = 0,1$ in stopband. As a result, the Hankel matrix can be rewritten as (18), where $L_{p,i}$ and $L_{s,i}$ are the number of frequency sampling points in passband and stopband, respectively. Therefore, only the elements $H_i(1, v)$ and $H_i(N_i, v)$, $1 \le v \le N_i$, at the first row and last row, respectively, required to be computed. The solution of such a system of equations can be solved using Cholesky or Levinson decomposition technique that requires only $O(N^2)$ complexity [11]-[13].

$$\begin{aligned} H_i(u,v) = -\frac{1}{2}\Bigg\{ &\sum_{l=1}^{L_{p,i}} \cos\left[\left(2(u+v) \mp 1/2\right)\omega_l\right] + \\ &\sum_{l=1}^{L_{s,i}} \cos\left[\left(2(u+v) \mp 1/2\right)\omega_l \pm \pi/2\right]\Bigg\}, \end{aligned} \tag{18}$$

IV. SIMULATION RESULTS

In this section, a MATLAB software package is used to design two-channel QMF bank based on IIR all-pass filter having the same specifications as that of [7] for evaluating the performance of the proposed technique. The simulations are evaluated on the IBM PC with Intel Core 2 Duo 3.16GHz and 2.09GHz CPU with 1GB RAM. The design performance is evaluated in terms of peak stopband ripple (PSR) of low-pass analysis filter $H_0(e^{j\omega_l})$, maximal variation of phase response (MVPR) and maximal variation of group delay (MVGR) in $M(e^{j\omega_l})$, and maximal variation of QMF bank response (MVFBR), which are defined as follows:

$$PSR = 20\log_{10}\left(\max_{\omega_l \in [\omega_s,\, \pi]} H_0(e^{j\,\omega_l})\right) \tag{19}$$

$$MVPR = \max_{\omega_l \in [0,\, \pi]} \left| Arg\left\{M\left(e^{j\omega_l}\right)\right\} + (2N_0 + 2N_1 + 1)\omega_l \right| \tag{20}$$

$$MVGD = \max_{\omega_l \in [0,\, \pi]} \left| GD\left\{M\left(e^{j\omega_l}\right)\right\} - (2N_0 + 2N_1 + 1) \right| \tag{21}$$

$$MVFBR = \max_{\omega_l \in [0,\, \pi]} \left(20\log_{10}\left| M\left(e^{j\omega_l}\right) - \frac{1}{2} e^{-j(2N_0 + 2N_1 + 1)\omega_l} \right| \right) \tag{22}$$

where $Arg(\bullet)$ represents the designed phase and $GD(\bullet)$ represents the group delay response.

Example: Two real IIR all-pass filters $A_0(z^2)$ and $A_1(z^2)$ with orders $N_0 = 3$ and $N_1 = 2$, low-pass analysis filter $H_0(z)$ with passband edge frequency $\omega_p = 0.4\pi$ and stopband edge frequency $\omega_p = 0.6\pi$ are designed. The frequency sampling points is set $L_i = 8(N_i + 1)$. The resulting magnitude responses of low-pass and high-pass analysis filters are shown in Fig.2. Figure 3 shows the phase error responses compared with Lawson's method [6] and Lee's method [7]. Evidently, the designed phase error response is smaller than those of methods [6] and [7] in most of the frequency band. The overall QMF bank magnitude error responses are also compared in Fig. 4. As seen from this figure, the proposed method achieves better performance compared with Lawson's method [6] and Lee's method [7], except two lobe bands. The design performance comparison with methods of [6] and [7] are illustrated in Table I. It is clear that the design MVPR, MVGR and MVFBR values are superior to those of methods [6] and [7]; however, the PSR value is slightly larder than [6] and [7].

Simulation results indicate that the proposed method provides better performance than the techniques [6] and [7], except the PSR value. Additionally, by using some trigonometric properties, the system of linear equation associated matrix can be further formulated as a Toeplitz-

TABLE I
DESIGN PERFORMANCE COMPARISON

	Proposed	Lawson [6]	Lee and Yang [7]
PSR(dB)	-16.6959	-18.05	-19.97
MVPR(rad)	0.2023	0.3031	0.2060
MVGR	1.3873	2.1613	1.4980
MVFBR(dB)	-19.9138	-16.42	-19.76

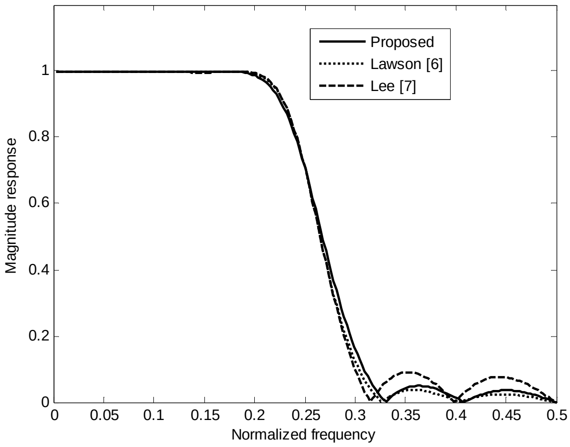


Fig. 2 Magnitude responses of low-pass analysis filter.

plus-Hankel matrix. Therefore, the optimal filter coefficients can be solved efficiently without computing a matrix inversion. Therefore, we concluded that the proposed technique not only achieves good performance but also with effectiveness.

V. CONCLUSION

This paper exploited the trigonometric properties to the design of two-channel quadrature mirror filter banks based on IIR all-pass filters. The all-pass filter design can be formulated as a system of linear equations solving problem. Using trigonometric properties, the linear system associated matrix can be formulated as a Toeplitz-plus-Hankel matrix expression such that the optimal filter coefficients can be designed efficiently without solving matrix inversion. Computer simulations have shown the proposed method achieves good performance and with effectiveness.

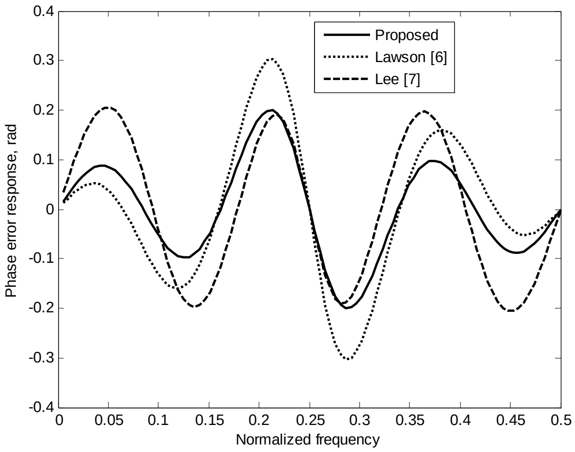


Fig. 3 Phase error responses compared with different methods.

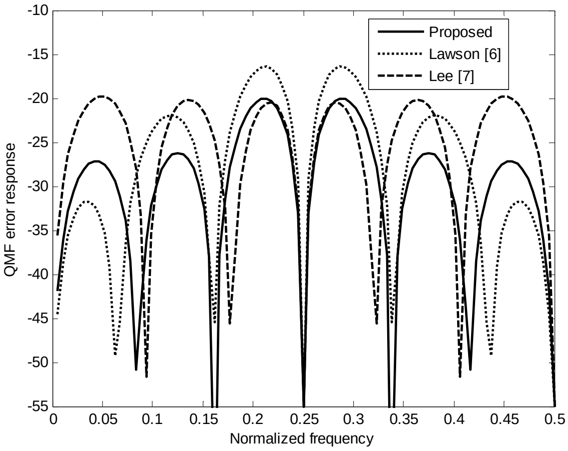


Fig. 4 QMF magnitude error responses compared with different methods.

REFERENCES

[1] M. J. T. Smith and S. L. Eddins, “Analysis/synthesis techniques for subband image coding,” *IEEE Trans. Acoust. Speech Signal Process.*, vol. 38 no. 8, pp. 1446-1456, Aug, 1990.

[2] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall, Englewood Cliffs, NJ, 1993.

[3] C. Herley, and M. Vetterli, “Wavelet and recursive filter banks,” *IEEE Trans. Signal Process.*, vol. 41, no. 8, pp. 2536-2566, Aug. 1993.

[4] S. S. Lawson, “Direct approach to design of PCAS filters with combined gain and phase specification,” *IEE Proc. Vis. Image Signal Process.*, vol. 141, no. 3, pp. 161-167, 1994.

[5] X. Zhang, T. Muguruma, and T. Yoshikawa, “Design of orthonormal symmetric wavelet filters using real all-pass filters,” *Signal Processing*, vol. 80, pp. 1551-1559, 2000.

[6] S. S. Lawson, and A. Klouche-Djedid, “Technique for design of two-channel approximately linear phase QMF filter bank and its application to image compression,” *IEE Proc. Vis. Image Signal Process.*, vol. 148, no. 2, pp. 85-92, April, 2001.

[7] J. H. Lee, and Y. H. Yang, “Design of two-channel linear-phase QMF banks based on real IIR all-pass filters,” *IEE Proc. Vis. Image Signal Process.*, vol. 150, no. 5, pp. 331-338, Oct. 2003.

[8] S. S. Kidambi, “Weighted least-squares design of recursive all-pass filters,” *IEEE Trans. Signal Process.*, vol. 44, pp. 1553-1557, June 1996.

[9] L. C. Su, Y. D. Jou, and F.K. Chen, “An efficient least-squares design of IIR all-pass filter,” in *Proc. IEEE TENCON 2009*, 23-26 Nov. Singapore 2009.

[10] G. Stancic, and S. Nikolic, “Digital linear phase notch filter design based on IIR all-pass filter application,” *Digital Signal Processing*, vol. 23, issue 3, pp.1065-1069, May 2013.

[11] S. Kidambi, “An efficient weighted least-squares design of linear-phase nonrecursive filters,” *IEEE Trans. Circuits and Systems-II*, vol. 42, no. 5, pp. 359-361, May 1995.

[12] G. A. Merchant and T. W. Parks, “Efficient solution of a Toeplitz-plus-Hankel coefficient matrix system of equations,” *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 30, pp. 40-44, Feb. 1982.

[13] P. Delsarte and Y. V. Genin, “The split Levinson algorithm,” *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 24, pp. 470-478, June 1986.