International Journal of Science and Engineering
http://140.133.4.61/IJSE/index.html
International Journal of Science and Engineering Vol.3, No.2 (2013): 17-23

Special Issue

A Case Study of Remedial Instruction of a Sixth Grader in Solving of Proportionality Problems in a Dynamic Multiple Representation Computer Environment

Ching-chih Li[1] and Che-jen Hsieh[2,*]

[1]Tainan Municipal Heshun Elementary School

[2]National Tainan Institute of Nursing

Abstract—We selected a sixth grade student to participate in this study. Prior to remedial instruction, the subject did not understand the concept of proportionality and used ratio formulas excessively to solve problems without any meaning on it. In addition, he could not employ multiple representations to explain his solutions. Using an electronic whiteboard and Geometer's Sketchpad (GSP) to design the dynamic multiple representations for proportional scenarios in remedial instruction induced the subject has better understanding of proportions as well as a better grasp of the linear relationship between two variables. In the final test, the subject was able to use multiple representations to solve problems and check his calculations, therefore increasing problem solving accuracy.

Index Terms: proportionality problems, multiple representations, Geometer's Sketchpad

I. Introduction

Proportionality problems are extensively applied in daily life and furthermore in science and technology. Learning concepts of proportionality is essential. Common proportionality problems include density, concentration, speed, and scale. The acquisition of concepts of ratio and proportionality are also an important benchmark in junior mathematics and serve as foundation to higher mathematics (calculus, trigonometry) [1]. Thus the importance of proportionality in the elementary school curriculum can be clearly seen. Instruction of proportionality, however teachers have often overemphasized pure numerical calculation and that the product of the extremes equals that of the means. This results in students solving problems mechanically. When the students encounter problems in the application of proportionality, they have difficulty solving them due to represent the variations and relations of the quantities. Deficiencies in present day education on proportionality come from the lack of teaching on visualization and real thematic scenarios; a graphical approach is often neglected in numerical approaches. In solving the part-whole problems, with the help of instruction of visual representation, students can both understand the problems and the process of solving the problems. The students are able to find not only the cues of solving problems but also the parts and whole relationship[2]. Comprehension based on actual experience or actual practice is beneficial to problem solving when students are taught proportionality problems in intensive quantities [3]. However, while previous studies have been conducted on hierarchical models and problem-solving strategies; few studies drawn attention to students' common daily life experience. Ratio sense can only be enhanced by practicing numerous types of problems or through the students' own experiences. Nevertheless, measuring or experiencing proportionality poses a challenge for traditional teaching approaches with the static curriculum; how to employ more precise teaching of intensive quantity problems, such as speed or concentration, in the classroom is also an art in itself. Expressing the abstract concepts and variables in proportions is also the bottleneck for many students. Young children cognitive development on variation of variable and using the variable to express the proportional relationship is still insufficient. This leads to the negative attitude of many students when facing proportionality. How can teachers intrigue their students into learning with proportionality reasoning? How can proportionality be more specifically represented to students? These are issues that demand immediate attention.

Teaching proportionality involves helping student to find the relationships of variables and designing the scenarios; computer-aided methods can be advantageous in this aspect [4]. Geometer's Sketchpad (GSP) provides precise compass and straightedge constructions. The changes of the two variables (two sets of terms) can be easily observed with the control of the action buttons, and the required covariance process is achieved by operation of the dynamic graphics. Simultaneously, GSP provides tables and coordinate graphs, which are useful external representation tools in teaching proportionality. Using an electronic whiteboard to handle proportionality problems gives users freedom in operation and makes the simulations more realistic. Students can freely

*Corresponding author: chejenhsieh@mail.ntin.edu.tw
DOI：10.6159/IJSE.2013.(3-2).04

manipulate the objects and feel like they are actually touching them, increasing the interactive learning experience. Many approaches can be employed in the representation of ratios and proportions, such as figures, tables, coordination graphs, figures, text, and symbols. If GSP can supply a complete dynamic visual environment for students to observe, control external representations, and connect with the representations, proportions will have more meaning to the students; conceptualizing abstract proportions will become effortless for them. Therefore the subject of this study used an electronic whiteboard and with the proportionality teaching materials designed by GSP, he operated the dynamic visuals activities. The visual connections with the multiple representations assisted the subject to visualize the abstract covariance concepts. Initial investigation of the subject in this environment revealed the reasons for incorrect solutions in the past, achieved minor development in representation abilities, as well as providing comprehension of issues beneficial to the subject's learning and the subject's responses.

II. Literature Review

2.1 Research Related to Proportionality Problems

[5] deemed that ratio and proportion concepts comprised three mathematical factors:

1. Relative and absolute change: A ratio is the relative magnitude of one numerical value to another, and "relative" is the most crucial element of this concept.
2. Ratio sense: Ratio sense is the perception of proportionality, in that children can distinguish what situations contain proportionality and can be expressed with an appropriate ratio. Ratio sense is the basis of solving proportionality problems, but this concept has not been included in past curriculum.
3. Covariance and invariance: The quantities that form a ratio are covariant (change together) under identical circumstances, but the relationship between them is invariant. However, even if a student writes down a series of equivalent ratios for a given ratio, understanding the invariant relationship between the terms is still difficult for them.

Additionally, three factors influence the ability of students to solve proportion problems:

1. Student characteristics: [6] pointed out that the characteristics of students determine the strategies they choose for problem solving; students with different characteristics will exhibit different performance in solving proportionality problems.
2. Numerical characteristics: Students are more skilled at proportionality problems with integral ratios and possess weaker ratio sense for non-integral ones [7]. The difficulty of proportionality problems also increases with the magnitude of the numerical values; solutions become more complicated and answers are difficult to obtain [8].
3. Problem content: When students are solving problems, problems with different context (semantics) affect the problem cognition and adopted solutions of students [6]. An integration of domestic and overseas classification methods [6]-[7] categorized proportionality problems into five types:

(1) Exchange: where two units of objects have the same value due to a certain reason and can be interchanged.

(2) Associated Sets: where two quantities have no apparent association until the problem specifies that they are proportional.

(3) Part-Part-Whole: where a set (whole) is comprised of two or more subsets (part) and the set and subset, or whole and part, are proportional.

(4) Well-chunked measures: where an intensive quantity, which is already known, is newly created from the proportional relationship of two extensive quantities, for example, density and speed.

(5) Stretchers and Shrinkers: where two quantities have a certain ratio, and one will increase or decrease with the other.

[6] discovered that in the latter two categories, students could not create concrete representation of the solutions with graphs or other approaches, and therefore many used lower level strategies to solve problems.

2.2 Research Related to Multiple Representation

[9] defined representation as "the external and therefore observable embodiments of students' internal conceptualizations" and divided representations into experience-based "scripts", manipulatable models, pictures or diagrams, spoken languages, and written symbols. Multiple representations are the expressions of the same conceptual structure with multiple types of representations [10]. However, textbook teaching nowadays often only provides students with single representations for math concepts; when translated to multiple representations, students often have issues[11].

This study adopted the cognitive schema theory proposed by [12] to arrange the levels of multiple representations of proportionality. She inferred that in the process of conception, operational concepts are conceived before structural concepts, though structural concepts are at a higher level than operational concepts. Converting computational operations to abstract objects is accomplished in three stages: (a) interiorization: in this stage, the learner gets acquainted with the process by object operation and induces new concepts or mental images; (b) condensation: in this phase, complex operations become a whole, and the learner no longer considers the procedures; (c) reification: this stage can be regarded as the transfer of the concept so that the learner can immediately view a familiar object from a whole new point of view.

III. Technical Work Preparation

3.1 Study Design

To better understand the learning progress of the subject as regards proportionality concepts and representation ability through use of computer-aided dynamic teaching materials, we needed first to understand the learning status and attitude of the subject prior to case study. The purposes of the teaching materials in this study were to improve connectivity between the mathematical elements and graphs of proportions and written symbol representations as well as assist the subject in correct grasp of concepts of proportionality. A sixth grader was selected for pretest and received five lessons on proportionality. The lessons were designed according to the problem types that the student answered incorrectly on the pretest, each with a different proportionality scenario. After the instruction experiment, a post-test was conducted to investigate changes in the subject's concepts of proportionality and representation ability after engaging in learning with the computer-aided dynamic teaching materials.

3.2 Study Subject

1. Expressing the Mathematical Elements of Proportions
   (1) Relativity and absoluteness: The subject focused more on the absoluteness of two quantities and had no concept of relativity.
   (2) Ratio sense: The subject could classify proportions such as exchange, well-chunked measures, and stretchers and shrinkers, but not completely.
   (3) Invariance and covariance: The subject grasped the concept of covariance in two quantities and a sense of invariance, but could not fully describe it.
2. Presenting Multiple Representations of Proportions
   (1) Graphs: The subject could draw a graph after deriving the numerical values, but could not solve proportionality problems directly from a graph or explain the relationship of proportionality. The subject was at the interiorization stage of this representation.
   (2) Tables: The subject was familiar with this method of representation, but the steps he used were too complicated. Thus the subject was still at the interiorization stage of this representation.
   (3) Proportion (symbol): The subject displayed considerable use of the equivalence of the extreme and mean products, but did not know the reason for the equivalence. However, with regard to the multiples of proportions, the subject was at the condensation stage.
   (4) Direct proportionality graph: This term was unfamiliar to the subject; therefore the subject was not at the interiorization stage yet.
   (5) Semantics: The subject could only explain some of the representations of proportionality, and so the subject was not at the interiorization stage of this representation.

The subject showed extensive ability to use written proportions for representation as a problem-solving strategy; he could not fully explain scenarios and lacked concepts of proportionality or direct proportionality graphs. This study hoped to increase the number of skills and representations that the subject could easily use after understanding ratio formulas with the assistance of the GSP scenarios.

3.3 Program Design

1. Principles of Program Design

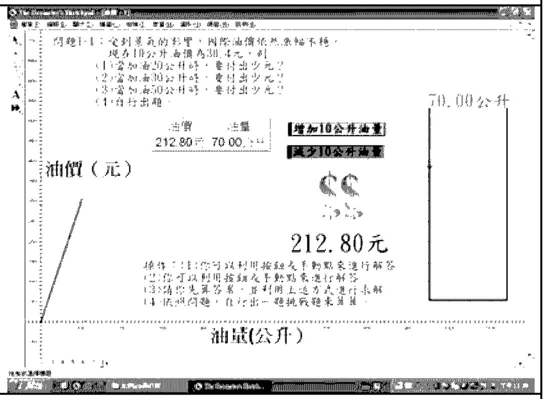
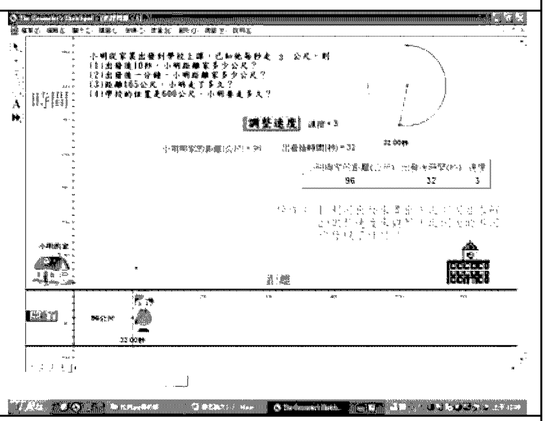
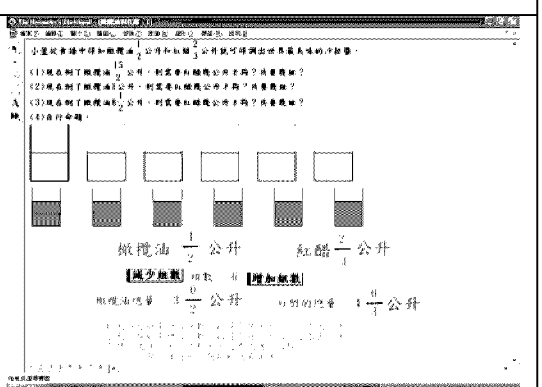
The greatest advantage of employing GSP for design is the dynamic connections between text and the other representations. The mathematical elements of proportions are blended into the scenarios and the abstract text problems can be swiftly displayed on the screen in a dynamic form. Simultaneously, GSP gives immediate feedback on the answers; changing the numerical values immediately renews the graphs and tables, from which students can generate stronger connection with the representations and further develop more problem-solving strategies. The features of the dynamic computer-aided teaching materials in this study are shown below:

Table 1
Explanation of Proportional Mathematical Elements in a Computer Environment

Mathematical Elements of Proportions	**Program Design**
Enhancing ratio sense	Five examples of proportionality were designed to enhance the subject's ratio sense and cognition of multiplying and dividing scenarios.
Focusing on use of relativity	An interactive example was designed in which the subject could adjust the numerical values and graphs and easily observe the relativity of two proportional quantities.
Emphasizing covariance and invariance	An adjustable object graph was designed to allow the subject to discover the covariance and invariance of the numerals from the multiples in the table and the problem-solving process.

Table 2
Design of Multiple Representations in a Computer Environment

Type of Representation	Proportional Multiple Representation	Program Design
Written symbols	Text	Problem scenario and handling steps
Graphs or Tables	Graph scenario	Generation of text-graph connection from changes in numerical values and graphs.
	Tables	Can be freely manipulated and the numbers changed. Ratio concept can be understood from the numerical models.
	Graph of Direct Proportionality	Graphs of direct proportionality can be drawn to understand how they are produced and to observe the corresponding relationship between numerical values and graphs.
Concrete manipulation		Graphs and numerical values can be manipulated by buttons.

Proportional Scenario	Exchange Scenario
Teaching Material Diagram	
Illations of Steps	User can infuse a10- unit oil each time by clicking the button. The oil quantities in tank are linked with the paid money. The data are collected automatically as a two-column table and a coordinate graph. User can paid their attention to the proportional activity without any action of measuring and drawing.
Proportional Scenario	Well-chunked Measures Scenario
Teaching Material Diagram	
Illations of Steps	The user can move the child icon to the left by clicking the button and analyze the relationship of tabulation values ; its coordinate graph.
Proportional Scenario	Associated Sets Scenario
Teaching Material Diagram	
Illations of operation	To click the buttons for changing the sets of cups, the correspondent values will follow to change .

2. Diagrams of Dynamic Computer-Aided Teaching Materials

Table 3
Diagrams of Proportional Mathematical Elements in a Computer Environment: Enhancing Ratio Sense (Diagrams of Five Proportional Scenario Examples)

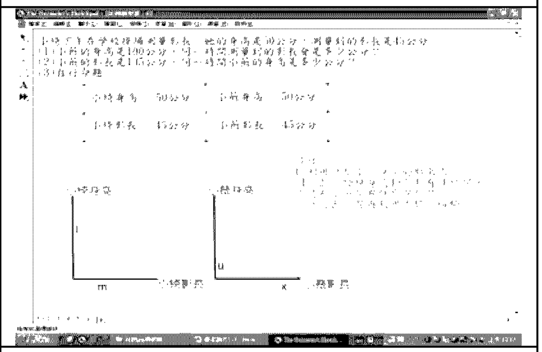
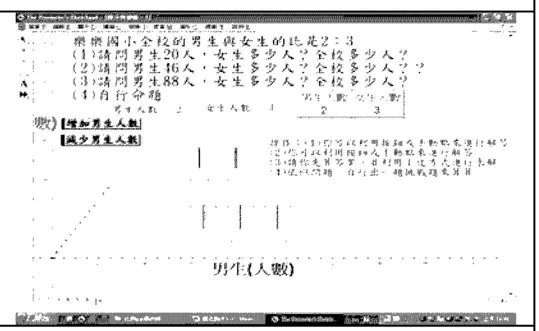
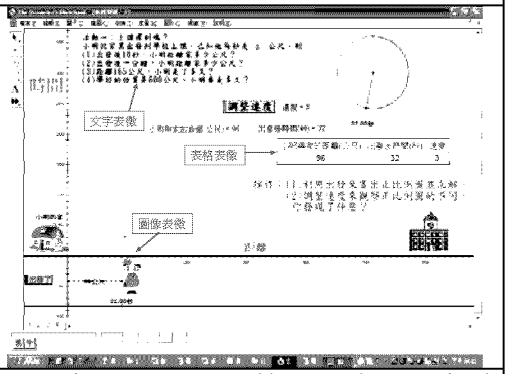
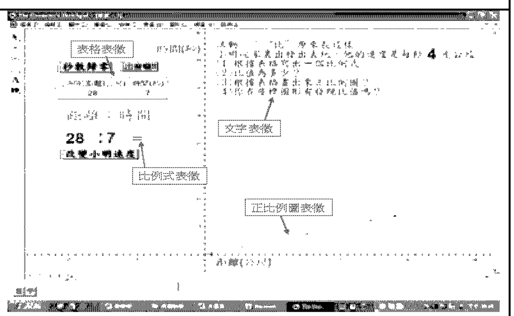
Proportional Scenario	Stretchers and Shrinkers Scenario
Teaching Material Diagram	
Illations of Steps	Drag the points on the left of triangle would cause the triangle on the right change. The user can tabulate the crucial values and find the rate of change.
Proportional Scenario	Part-part-whole Scenario
Teaching Material Diagram	
Illations of Steps	To click the button for changing the numbers of person icons and then observed the variation on table and its coordinate graph.

Table 4
Connection of multi- representations

Representation Connection	Connection Between Graphs, Text, and Table Numerals
Teaching Material Diagram	
Re-presentations Connection	Connection Between Text, Table Numerals, Proportional Expressions, and Proportional Graphs
Teaching Material Diagram	

IV. Result Analysis and Discussion

4.1 Subject Worksheet Records During Instruction

During the instruction, the subject discovered the relationship between scenarios of proportionality, numerals, and graphs and generalized a conclusion by manipulating the graphs, changing the numerals, and observing the changes.

4.2The Subject Difficulties in Solving the Proportional problems

(1). The subject employ the product of the extremes equals that of the means strategy excessively. He often misplace the values just according to the surface appearance on the problem. He cannot even find out the mistakes.
(2). He has seldom experience on the variation of values and cannot explain the proportionality graphically.
(3). He cannot solve the unknown value is not integer in ratio problems. While the problems occurrence are unfamiliar to him, he can only exhaust all the numbers instead of using the proportional reasoning to pursuit the solution.

4.3 Circumstances and Analysis of the Subject's Answers Pretest and Post-test

1. Expressing the Mathematical Elements of Proportions

(1) Relativity and Absoluteness

Problem: Chiyo and Wei are measuring shadows on the school playground in the afternoon. Chiyo's height is 150 cm and her shadow is 90 cm long. If Wei is 160 cm tall, how long is her shadow?
(Posttest)
T: Problem 5 of Task 4. If Chiyo used to be 150 cm tall and is now 160 cm, and Wei used to be 140 cm tall and is now 150 cm, whose height increased more?
S: The same!
…………………………………………
T: Then who grew faster?
S: First calculate how many centimeters they grew, then divide them by time and you get how fast they grew.

Analysis: After instruction, the subject was still more focused on absoluteness than on relativity. However, he now had an initial concept of relativity and did not emphasize the relationship between the two quantities. The post-test interview shows that the subject compared the growth rates by using time as well as height.

(2) Ratio Sense

(Posttest)
T: Which problems do you think are not proportionality problems in this test paper?
S: All of them are.
…………………………………………..
T: What kinds of things in life do you think are proportionality problems?
S: Making things bigger or smaller, putting two different things together… and problems that give you the whole and want you to find the part…do concentration and speed count?
T: Anything else that is a proportionality problem?
S: Like gas price problems, how much a liter is, and how many liters a dollar can buy…no more (shakes his head).
T: Why do you think concentration and speed count?
S: Because they are both specific values of ratios.

Analysis: The subject concluded more types of proportionality problems and placed more importance on multiples in proportionality problems. He acknowledged the following proportionality problems: exchange problems, well-chunked measure problems such as concentration and speed, and stretcher and shrinker problems. Therefore, the subject's ratio sense has improved immensely.

(3) Invariance and Covariance

Problem: The amount of food the pythons at the zoo get every day is determined by their length. Python A is 25 meters long and gets 10 eggs every day. If Python B is 15 meters long, how many eggs does it get?
(Posttest)
T: Number 8 of Task 4. Use a table to check your calculations. Can you explain what changes together and what doesn't?
S: The specific value doesn't change.
T: What changes at the same time?
S: The original ratio changes together.

Analysis: In the pretest the subject was very familiar with the concept of covariance in proportions. In the instruction, the subject gained further comprehension of invariance and covariance through table representation.

2. Presenting Multiple Representations of Proportions

(1)Problem-Solving Approach：Graph Method

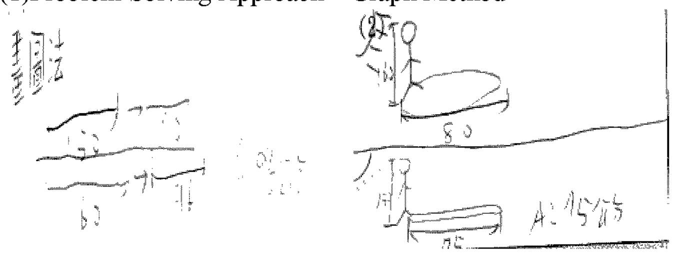


Figure 1 Graph Method on Pretest and Posttest

Analysis：He cannot solve the proportionality problem graphically in the pretest. With the help of dynamic computer design, the subject can represent the proportionality problem graphically and then solve the problem. After the instruction, the subject improved considerably in graph representation, achieving the condensation stage and being able to explain with illustrated graphs.

(2)Problem-Solving Approach：Table Method

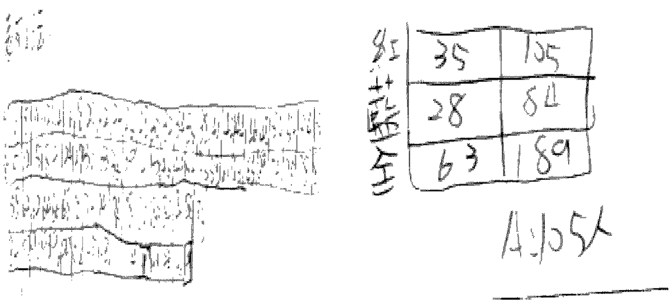


Figure 2 Table Method on Pretest and Posttest

Analysis：The subject exhausted all the numbers instead of using the proportional reasoning to find the solution. With the help of the dynamic computer design, the subject can really generate the proportionality and then find the right solution. The subject advanced to the condensation stage in this representation after instruction.

(3)Problem-Solving Approach：Proportion Method

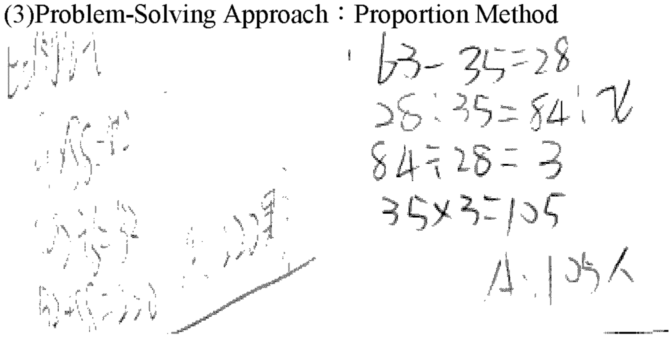


Figure 3 Proportion Method on Pretest and Posttest

Analysis：After instruction the subject no longer used the equivalence of extreme and mean products to solve problems; he instead used the multiples in proportions to explain and thus advanced to the condensation stage.

(4)Problem-Solving Approach：Direct Proportionality Graph Method

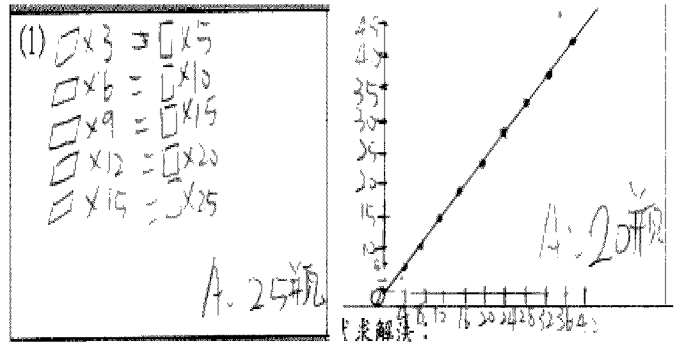


Figure 4 Direct Proportionality Graph Method on Pretest and Posttest

Analysis：The subject cannot draw a coordinate graph at the previous test. After the instruction, he can draw the graph. He represented the ratio relationship as a linear graph and solve the problem graphically. The subject employed correct application of direct proportionality graphs and could explain as well as solve problems. However, not every problem was answered correctly and so the subject was only in the interiorization stage.

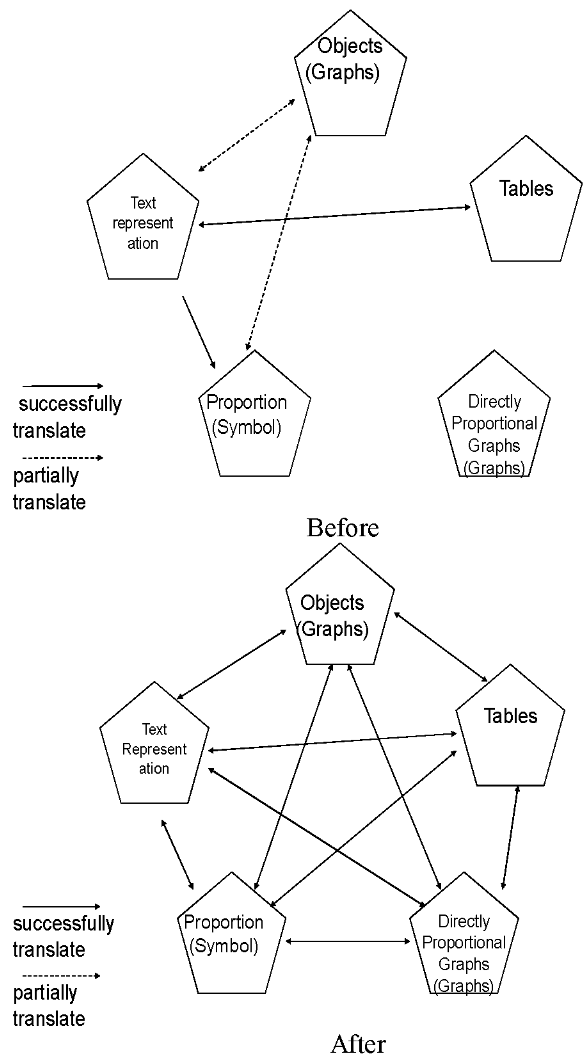


Figure 5 Subject Translation Approaches Before and After Instruction

Table 5
Comparison of Subject Methods in Obtaining Unknown Numbers in Proportional Scenarios Before and After Instruction

Problem Scenario	Exchange		Associated Sets		Part-part-whole		Well-chunked Measures		Stretchers and Shrinkers	
Test	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Object Graph Method	△	O	□	O	△	O	X	O	□	O
Table Method	△	O	△	O	△	O	△	O	△	O
Ratio Formula Method	□	O	□	O	□	O	□	O	□	O
Direct Proportionality Graph Method	X	O	X	O	X	O	X	O	X	O

Representation Ability: “O” = reification stage; “□” = condensation stage; “△” = interiorization stage; “X” = lack of concept

V. Conclusion and Recommendations

1. Progress of Concepts of Proportionality

(1) Relativity: The subject still made a point of using absoluteness more but was also starting to use relativity.

(2) Ratio sense: The subject’s ratio sense improved; he could confirm what scenarios were proportional and classify them.

(3) Invariance and covariance: From presentation of tables, the subject learned which quantities were covariant or invariant in ratio relationships and could apply them to proportions.

(4) Problem-solving strategies: The subject could use multiple representations to solve problems as well as employ them in explaining the relationships in ratios.

2. Change in Representation Abilities

(1) Graph method: Using a computer, the subject could accurately draw the graph of a proportional scenario.

(2) Table method: The subject could simplify a table and explain the proportional relationships in it.

(3) Ratio formula method: The subject could correctly solve a problem using proportions and explain the proportional relationships in it.

(4) Direct proportionality graph method: The subject could draw a correct direct proportionality graph and use it to explain the semantics.

(5) The instruction on proportionality was especially beneficial to the subject’s comprehension and analysis in the graph method, table method, ratio formula method, and direct proportionality graph method.

3. Recommendations for Further Study: This study only conducted computer remedial instruction on direct proportions and derived satisfactory results. Whether instruction aided by an electronic whiteboard can be applied to other mathematical units to enhance learning efficiency requires further research.

References

[1]Lamon, J. “Ratio and proportion: Cognitive foundations in unitizing and norming.” In G. Harel & J. Confrey (Eds.). The Development of Multiplicative Reasoning in the Learning of Mathematics (pp. 89-122). State University of New York Press,1994.

[2] Liu, C. T, Huang, K. S. and Sue, I. C. “Visual representation of problem solving strategy- taking example as solving the part-whole proportional problem.” Science Education Monthly. 329,2-7, 2010.

[3] Shen, M. H. & Liu, S. T. “Analyzing Literatures of Students’ Solving Proportion Problems-Suggestions for Curriculum and Instruction of Elementary Mathematics.” Research and Development in Science Education Quarterly, 27, 81-95,2002.

[4]National Council of Teachers of Mathematics. The principles and standards for school mathematics. Reston, VA: NCTM,2000.

[5] Lamon, J. “Ratio and proportion: Elementary didactical phenomenology.” In J. T. Sowder, & B. P. Schappelie (Eds),Providing a Foundation for Teaching Mathematics in the Middle Grades(pp.167-198). New York: State University of New York Press,1995.

[6] Lamon, J. “Ratio and proportion : connecting content and children’s thinking.” Journal for Research Mathematics Education. 24 (1), 41-61,1993. Doi: 10.2307/749385

[7] Yang, C. L. A Study of Detecting Senior Elementary Student in Solving. National Chiayi University Department of Education, unpublished master thesis, Chiayi,1998.

[8] Hart,K.M. Children's understanding of mathematics. London:John Murray,1981.

[9] Lesh, R., Post, T. & Behr M. “Representations and translations among representations in mathematics learning and problem solving.” Problems of representation in the teaching and learning of mathematics (pp 33-40). Hillsdale, NJ: Lawrence Erlbaum,1987.

[10] Tsay, J. R. & Tso, T. Y. “Cognitive Characteristics of Senior High School Students in Constructing Multiple Representations of Ellipse.” Chinese Journal of Science Education, 9(3), 281-297,2001.

[11] Hsieh, C. J. & Lin, J. K. “Dynamic and Visual Computer Activities Design for Learning Factors and Multiples in the Elementary School.” Journal of NUTN: Science and Engineering, 40(1), 23-45,2006.

[12] Sfard, “A. On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same Coin.” Educational Studies in Mathematics, 22, 1-36,1991. Doi: 10.1007/BF00302715

Biographies

Li Ching-chih completed a master degree at the field of mathematics education from National Tainan University. He is serving at Tainan Heshun Elementary School as a teacher, and a part-time mathematics instructor at National Tainan Institute of Nursing. He has published several articles on the field of learning mathematics concept with the computer.

Hsieh che-jen completed his PH. D from the University of Georgia. He is currently serving as associate professor at National Tainan Institute of Nursing. He has published more than 100 articles on the field of learning mathematics concept with the computer.