

A new approach for ranking fuzzy numbers

Jing-Ju Chiu¹ Bih-Sheue Shieh^{2,*}¹National Tainan Second Senior High School, No.125, Sec. 2, Beimen Rd., Tainan City 704, Taiwan

Tel.: +886-6-2514526 E-mail: jingjuc@yahoo.com.tw

²Department of Applied Mathematics, National University of Tainan 33, Sec. 2, Shu-Lin St., Tainan 700, Taiwan

Tel.: +886-6-2133111/ext. 559; fax: +886-6-3017131

E-mail: shieh@mail.nutn.edu.tw

ABSTRACT-- IN MANY FUZZY DECISION PROBLEMS, RANKING FUZZY NUMBERS IS AN IMPORTANT PROCEDURE. BECAUSE OF THE IMPERFECTION AND FUZZINESS OF THINKING, WE NEED A METHOD TO COMPARE FUZZY NUMBERS. SO FAR, A GREAT DEAL OF RESEARCH HAS BEEN OBTAINED. SOME OF THEM USE THE CENTROID POINT OF FUZZY NUMBERS AS AN INDEX TO RANK THE FUZZY NUMBERS. HAVING REVIEWED THE PREVIOUS RESEARCHES, WE FOUND THAT ALMOST EACH OF THEM TRIED TO USE A FORMULA TO RANK THE FUZZY NUMBERS. HOWEVER, MOST OF THEM HARDLY CAN RANK WITH INTUITION CONSISTENTLY IN ALL CASES. THEREFORE, A NEW METHOD OF RANKING FUZZY NUMBERS IS PROPOSED AND COMPENSATES FOR THESE SHORTCOMINGS. IT USES THREE INDICES TO DIFFERENTIATE BETWEEN FUZZY NUMBERS. THE REPRESENTATION LOCATION OF FUZZY NUMBER A IS A TOP PRIORITY. THE PLACE THAT THE LARGEST MEMBERSHIP FUNCTION GRADE HAPPENED IS THE SECOND. AND THEN IS S_A WHICH MEANS THE REMAINDER AREA IS THE LAST. FINALLY, EFFICIENCY AND SUPERIORITY OF THE PROPOSED METHOD ARE ILLUSTRATED BY SOME NUMERICAL EXAMPLES.

I. INTRODUCTION

In many fuzzy decision problems, ranking fuzzy numbers is an important procedure. Because of the imperfection and fuzziness of thinking, we need a method to compare fuzzy numbers.

So far, a great deal of research has been obtained. Some of them use the centroid point of fuzzy numbers as an index to rank the fuzzy numbers. Having reviewed the previous researches, we found that almost each of them tried to use a formula to rank the fuzzy numbers. However, most of them hardly can rank with intuition consistently in all cases.

In 1998, Cheng [2] gave a method to rank fuzzy numbers based on the distance between the centroid and the original point. Chu and Tsao [4] improved by proposing another method based on the area between the centroid point and original point. Xu and Wei [12] reformed the method given by Cheng [2] and Chu and Tsao [4]. But it is still unsolved. Wang and Liu [10] considered the relative importance of the centroid, and the relative dominance and priority weight, and then proposed a new way of ranking. However, the procedure of ranking is complicated and time-consuming.

Therefore, a new method of ranking fuzzy numbers is proposed and compensates for these shortcomings. Finally, efficiency and superiority of the proposed method are illustrated by some numerical examples.

II. PRELIMINARIES

Let U be a classical set of objects, called the universe, whose generic elements are denoted by x .

Definition 1.1.

Let U be a universe set. A fuzzy set A of U is defined by a membership function $\mu_A(x) \rightarrow [0, 1]$, where $\mu_A(x), \forall x \in U$, indicates the degree of x in A .

Definition 1.2.

A fuzzy subset A of U is normal if and only if $\sup_{x \in U} \mu_A(x) = 1$

Definition 1.3.

A fuzzy subset A of U is convex if and only if $\mu_A(\lambda x + (1-\lambda)y) \geq \min(\mu_A(x), \mu_A(y)), \forall x, y \in U, \lambda \in [0, 1]$

Definition 1.4.

A fuzzy set A is a fuzzy number if and only if A is normal and convex on U .

In this work, we consider the fuzzy numbers that have continuous membership functions.

Lemma 1.1[1].

The membership function μ_A of a fuzzy number A is defined by

$$\mu_A(x) = \begin{cases} \mu_A^L(x), & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \mu_A^R(x), & c \leq x \leq d \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

where μ_A^L, μ_A^R are two monotonic and continuous mapping from \mathbb{R} to the closed interval $[0, 1]$.

Definition 1.5.

A triangular fuzzy number A is a fuzzy number with a membership function μ_A defined by

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{c-x}{c-b}, & b \leq x \leq c \\ 0, & \text{otherwise} \end{cases}$$

A triangular fuzzy number A can be denoted as (a, b, c) (see Fig. 1.1).

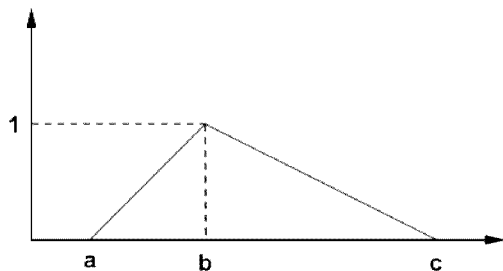


Fig.1.1

Definition 1.6.

A triangular fuzzy number $-A$ is a fuzzy number with a membership function μ_{-A} (see Fig. 1.2) defined by

$$\mu_{-A}(x) = \begin{cases} \frac{x+a}{b-a}, & -b \leq x \leq -a \\ \frac{c+x}{c-b}, & -c \leq x \leq -b \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.7.

A trapezoidal fuzzy number A is a fuzzy number with a membership function μ_A defined by

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \frac{d-x}{d-c}, & c \leq x \leq d \\ 0, & \text{otherwise} \end{cases}$$

A trapezoidal fuzzy number A can be denoted as (a, b, c, d) (see Fig. 1.3).

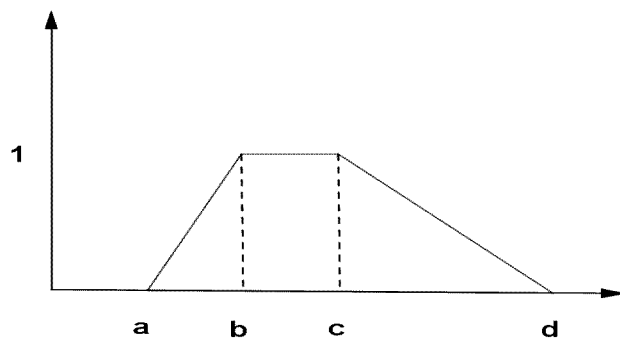


Fig.1.3

Definition 1.8.

Given a fuzzy set A and any number $\alpha \in [0, 1]$, the α -cut ${}^\alpha A$ is the crisp set ${}^\alpha A = \{x | \mu_A(x) \geq \alpha\}$.

Lemma 1.2[1].

For every fuzzy set A,

$$A = \bigcup_{\alpha \in [0,1]} {}^\alpha A, \text{ where } {}^\alpha A = \alpha \cdot {}^\alpha A \text{ and } \cup \text{ denotes the standard fuzzy union.}$$

III. CENTROIDS OF FUZZY NUMBERS

Most of the methods for ranking fuzzy numbers depend on the centroids of fuzzy numbers. Therefore, we discuss the centroid of a fuzzy number in this section. We first introduce the centroid formula proposed by Wang [8]. This formula is adopted by most of the researchers. Then, we discuss an improved formula that proposed by Shieh [6]. Finally, the methods of a triangular fuzzy number and a trapezoidal fuzzy number are obtained.

Wang [8] suggested a pair of centroid formula (x_0, y_0) for a fuzzy number A by

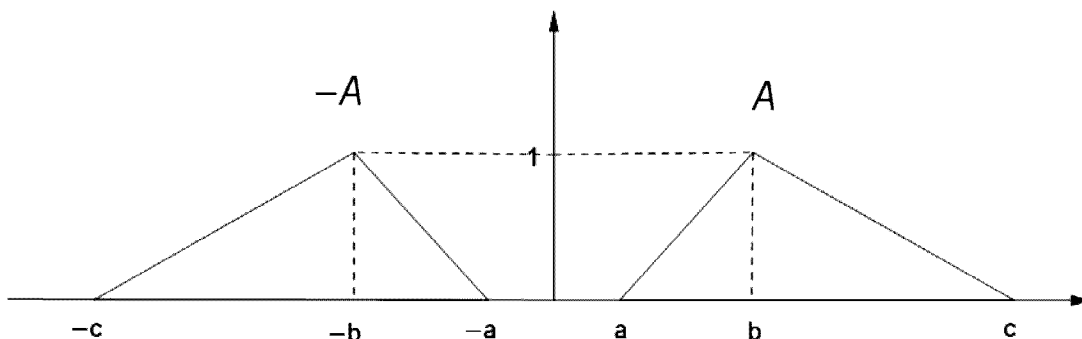


Fig.1.2



$$\begin{aligned} \bar{x}_0(A) &= \frac{\int_{-\infty}^{\infty} x \cdot \mu_A(x) dx}{\int_{-\infty}^{\infty} \mu_A(x) dx} \\ &= \frac{\int_a^b x \cdot \mu_A^L(x) dx + \int_b^c x \cdot \omega \cdot dx + \int_c^d x \cdot \mu_A^R(x) dx}{\int_a^b \mu_A^L(x) dx + \int_b^c \omega \cdot dx + \int_c^d \mu_A^R(x) dx} \end{aligned} \quad (2)$$

$$\bar{y}_0(A) = \frac{\int_0^1 y \cdot ((\mu_A^R)^{-1}(y) - (\mu_A^L)^{-1}(y)) dy}{\int_0^1 ((\mu_A^R)^{-1}(y) - (\mu_A^L)^{-1}(y)) dy} \quad (3)$$

where μ_A is defined as (1) with μ_A^L and μ_A^R are two strictly monotonic and continuous functions.

Shieh [6] extended Wang's formulae to a more general case defined by

$$\bar{x}_0 = \frac{\int_{-\infty}^{\infty} x \cdot \mu_A(x) dx}{\int_{-\infty}^{\infty} \mu_A(x) dx} \quad (4)$$

$$\bar{y}_0 = \frac{\int_0^{\omega} \alpha \cdot |\alpha A| d\alpha}{\int_0^{\omega} |\alpha A| d\alpha}, \quad (5)$$

where $|\alpha A|$ is the length of the α -cut

${}^\alpha A, 0 < \alpha \leq 1$

To all intents and purposes Shieh's centroid formulae are the same with Wang's formulae. But they don't need membership function invertible. It is more nature and simple way to determine the centroid point of fuzzy number. Most important, it is available in any cases.

Lemma 3.1.

If $(\mu_A^R)^{-1}$ and $(\mu_A^L)^{-1}$ exist then formula (3) and (5) are the same.

Proof.

Note that

$$(\mu_A^R)^{-1}(y) = \sup\{x | \mu_A^R(x) = y\} \text{ and}$$

$$(\mu_A^L)^{-1}(y) = \inf\{x | \mu_A^L(x) = y\}.$$

Since $\mu_A^R(x)$ is decreasing, $(\mu_A^R)^{-1}(y) = \sup\{x | x \in {}^\alpha A\}$.

Similarly, since $\mu_A^L(x)$ is increasing

$$(\mu_A^L)^{-1}(y) = \inf\{x | x \in {}^\alpha A\}.$$

By definition

$${}^\alpha A = \{x | \mu_A(x) \geq \alpha\} = [(\mu_A^L)^{-1}(\alpha), (\mu_A^R)^{-1}(\alpha)], \text{ which}$$

is a closed interval, for all $0 < \alpha \leq 1$.

Therefore, $|\alpha A| = (\mu_A^R)^{-1}(\alpha) - (\mu_A^L)^{-1}(\alpha)$ and formula (3) coincides with formula (5).

Remark 3.1.

If A is a triangular fuzzy number denoted as (a, b, c),

$$\text{then } {}^\alpha A = \left[a + (b-a) \cdot \alpha, c - (c-b) \cdot \alpha \right]$$

(see Fig.3.1),

$$\text{and } |{}^\alpha A| = (c-a) + (a-c)\alpha.$$

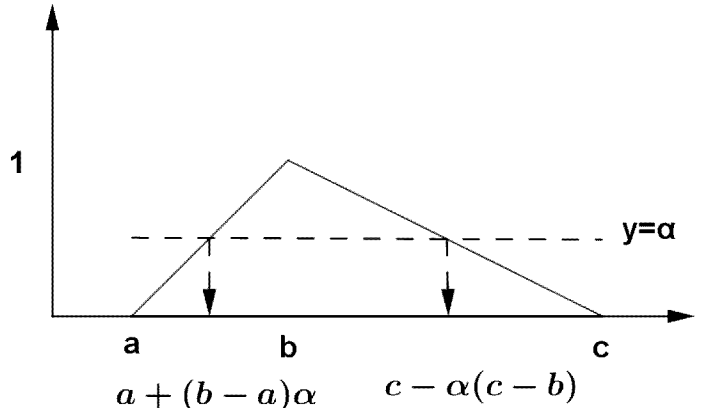


Fig.3.1

Thus,

$$\begin{aligned} & \int_0^1 |{}^\alpha A| d\alpha \\ &= \int_0^1 [(c-a) + (a-c)\alpha] d\alpha \\ &= (c-a)\alpha + \frac{1}{2}(a-c)\alpha^2 \Big|_0^1 \\ &= (c-a) + \frac{1}{2} \cdot (a-c) \\ &= \frac{1}{2}(c-a) \\ &= \int_{-\infty}^{\infty} A(x) dx. \end{aligned}$$

Remark 3.2.

If A is a trapezoidal fuzzy number denoted as (a, b, c, d), then

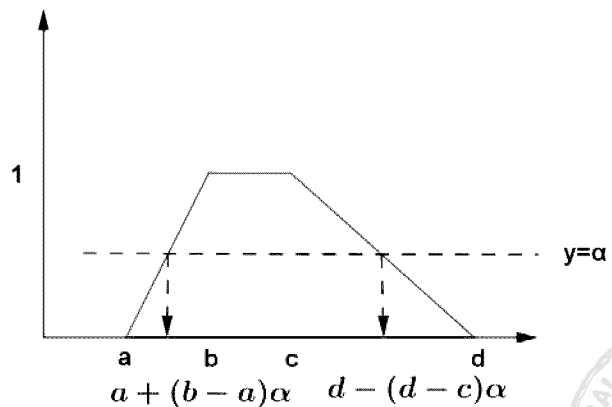


Fig.3.2



$${}^{\alpha}A = \left[\mathbf{a} + (\mathbf{b} - \mathbf{a}) \cdot \alpha, \mathbf{d} - (\mathbf{d} - \mathbf{c}) \cdot \alpha \right] \quad (\text{see Fig.3.2),}$$

$$\text{and } |{}^{\alpha}A| = (\mathbf{d} - \mathbf{a}) + (\mathbf{c} - \mathbf{d} - \mathbf{b} + \mathbf{a}) \cdot \alpha.$$

$$\int_0^1 |{}^{\alpha}A| d\alpha = \int_0^1 [(\mathbf{d} - \mathbf{a}) + (\mathbf{c} - \mathbf{d} - \mathbf{b} + \mathbf{a}) \cdot \alpha] d\alpha$$

$$\text{Thus, } = (\mathbf{d} - \mathbf{a})\alpha + \frac{1}{2}(\mathbf{c} - \mathbf{d} - \mathbf{b} + \mathbf{a}) \cdot \alpha^2 \Big|_0^1$$

$$= (\mathbf{d} - \mathbf{a}) + \frac{1}{2} \cdot (\mathbf{c} - \mathbf{d} - \mathbf{b} + \mathbf{a}).$$

$$= \frac{1}{2}(\mathbf{c} - \mathbf{b} + \mathbf{d} - \mathbf{a})$$

$$= \int_{-\infty}^{\infty} A(x) dx.$$

By formulae (4) and (5), we can get the centroids of a triangular and a trapezoidal fuzzy numbers, respectively.

Lemma 3.2.

If A is a triangular fuzzy number denoted as (a, b, c), then $\bar{x}_0(A) = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ and $\bar{y}_0(A) = \frac{1}{3}$.

Proof.

Notably,

$$\begin{aligned} & \int_{-\infty}^{\infty} x \cdot \mu_A(x) dx \\ &= \int_a^b \left(\frac{x-a}{b-a} \right) \cdot x dx + \int_b^c \left(\frac{c-x}{c-b} \right) \cdot x dx \\ &= \frac{1}{b-a} \left[\frac{1}{3} x^3 - \frac{1}{2} a x^2 \Big|_a^b \right] + \frac{1}{c-b} \left[\frac{1}{2} c x^2 - \frac{1}{3} x^3 \Big|_b^c \right] \\ &= \frac{1}{3}(\mathbf{b}^2 + \mathbf{a}\mathbf{b} + \mathbf{a}^2) - \frac{1}{2}(\mathbf{a}\mathbf{b} + \mathbf{a}^2) + \frac{1}{2}(\mathbf{c}^2 + \mathbf{b}\mathbf{c}) - \frac{1}{3}(\mathbf{c}^2 + \mathbf{b}\mathbf{c} + \mathbf{b}^2) \\ &= \frac{1}{6}(-\mathbf{a}^2 + \mathbf{c}^2 + \mathbf{b}\mathbf{c} - \mathbf{a}\mathbf{b}) \\ &= \frac{1}{6}(\mathbf{c} - \mathbf{a})(\mathbf{a} + \mathbf{b} + \mathbf{c}), \end{aligned}$$

and

$$\begin{aligned} & \int_{-\infty}^{\infty} \mu_A(x) dx \\ &= \int_a^b \left(\frac{x-a}{b-a} \right) dx + \int_b^c \left(\frac{c-x}{c-b} \right) dx \\ &= \frac{1}{2}(\mathbf{b} + \mathbf{a}) - \mathbf{a} + \mathbf{c} - \frac{1}{2}(\mathbf{c} + \mathbf{b}) \\ &= \frac{1}{2}(\mathbf{c} - \mathbf{a}). \end{aligned}$$

Thus,

$$\bar{x}_0 = \frac{\int_{-\infty}^{\infty} x \cdot \mu_A(x) dx}{\int_{-\infty}^{\infty} \mu_A(x) dx} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}).$$

Since

$${}^{\alpha}A = \left[\mathbf{a} + (\mathbf{b} - \mathbf{a}) \cdot \alpha, \mathbf{c} - (\mathbf{c} - \mathbf{b}) \cdot \alpha \right],$$

$$|{}^{\alpha}A| = (\mathbf{c} - \mathbf{a}) + (\mathbf{a} - \mathbf{c})\alpha.$$

Thus,

$$\begin{aligned} & \int_0^1 |{}^{\alpha}A| \cdot \alpha d\alpha \\ &= \int_0^1 [(\mathbf{c} - \mathbf{a})\alpha - (\mathbf{c} - \mathbf{a})\alpha^2] d\alpha \\ &= \left[\frac{1}{2}(\mathbf{c} - \mathbf{a})\alpha^2 - \frac{1}{3}(\mathbf{c} - \mathbf{a})\alpha^3 \right]_0^1 \\ &= \frac{1}{6}(\mathbf{c} - \mathbf{a}), \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 |{}^{\alpha}A| d\alpha \\ &= \int_0^1 [(\mathbf{c} - \mathbf{a}) - (\mathbf{c} - \mathbf{a})\alpha] d\alpha \\ &= (\mathbf{c} - \mathbf{a})\alpha - \frac{1}{2}(\mathbf{c} - \mathbf{a})\alpha^2 \Big|_0^1 \\ &= \frac{1}{2}(\mathbf{c} - \mathbf{a}). \end{aligned}$$

Therefore,

$$\bar{y}_0(A) = \frac{\int_0^1 \alpha \cdot |{}^{\alpha}A| d\alpha}{\int_0^1 |{}^{\alpha}A| d\alpha} = \frac{1}{3}.$$

Lemma 3.3.

If A is a trapezoidal fuzzy number denoted as (a, b, c, d), then

$$\bar{x}_0(A) = \frac{1}{3} \left(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} - \frac{\mathbf{c}\mathbf{d} - \mathbf{a}\mathbf{b}}{(\mathbf{c} + \mathbf{d}) - (\mathbf{a} + \mathbf{b})} \right) \quad \text{and}$$

$$\bar{y}_0(A) = \frac{1}{3} \left(1 + \frac{\mathbf{c} - \mathbf{b}}{(\mathbf{c} + \mathbf{d}) - (\mathbf{a} + \mathbf{b})} \right).$$

Proof.

Notably,

$$\begin{aligned} & \int_{-\infty}^{\infty} x \cdot \mu_A(x) dx \\ &= \int_a^b \left(\frac{x-a}{b-a} \right) \cdot x dx + \int_b^c x dx + \int_c^d \left(\frac{d-x}{d-c} \right) \cdot x dx \\ &= \frac{1}{b-a} \left[\frac{1}{3} x^3 - \frac{1}{2} a x^2 \Big|_a^b \right] + \frac{1}{2} x^2 \Big|_b^c + \frac{1}{d-c} \left[\frac{1}{2} d x^2 - \frac{1}{3} x^3 \Big|_c^d \right] \\ &= \frac{1}{6}(-\mathbf{a}^2 - \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{d}^2 - \mathbf{a}\mathbf{b} + \mathbf{c}\mathbf{d}), \end{aligned}$$



and

$$\begin{aligned} & \int_{-\infty}^{\infty} \mu_A(x) \\ &= \int_a^b \left(\frac{x-a}{b-a} \right) dx + \int_b^c 1 dx + \int_c^d \left(\frac{d-x}{d-c} \right) dx \\ &= \frac{1}{2}(-a-b+c+d). \end{aligned}$$

Therefore,

$$\bar{x}_0(A) = \frac{\int_{-\infty}^{\infty} x \cdot \mu_A(x) dx}{\int_{-\infty}^{\infty} \mu_A(x) dx} = \frac{1}{3} \left(a+b+c+d - \frac{cd-ab}{(c+d)-(a+b)} \right)$$

Since

$${}^\alpha A = \left[a + (b-a) \cdot \alpha, c - (c-b) \cdot \alpha \right],$$

$$|{}^\alpha A| = (c-a) + (a-c)\alpha.$$

Thus,

$$\begin{aligned} & \int_0^1 |{}^\alpha A| \cdot \alpha \cdot d\alpha \\ &= \int_0^1 [(d-a)\alpha + (c-d-b+a)\alpha^2] d\alpha \\ &= \left[\frac{1}{2}(d-a)\alpha^2 - \frac{1}{3}(c-d-b+a)\alpha^3 \right]_0^1 \\ &= \frac{1}{6}(-a-2b+2c+d), \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 |{}^\alpha A| d\alpha \\ &= \int_0^1 [(d-a) + (c-d-b+a)\alpha] d\alpha \\ &= \left[(d-a)\alpha + \frac{1}{2}(c-d-b+a)\alpha^2 \right]_0^1 \\ &= \frac{1}{2}(c+d-a-d). \end{aligned}$$

Therefore,

$$\bar{y}_0(A) = \frac{\int_0^1 \alpha \cdot |{}^\alpha A| d\alpha}{\int_0^1 |{}^\alpha A| d\alpha} = \frac{1}{3} \left(1 + \frac{c-b}{(c+d)-(a+b)} \right).$$

IV. RANKING FUZZY NUMBERS

Ranking fuzzy numbers is an important part of decision-making process. In this section, we discuss several of methods for ranking fuzzy numbers and point out their shorting.

4.1 Cheng's method [2].

Step 1. Find the centroid point (\bar{x}, \bar{y}) of fuzzy numbers.

Step 2. Calculate the distance index between the

centroid point (\bar{x}, \bar{y}) and original point.

$$\text{i.e., } R(A) = \sqrt{(\bar{x})^2 + (\bar{y})^2}$$

Step 3. Rank fuzzy numbers,

(1) if $R(A_i) < R(A_j)$, then $A_i < A_j$

(2) if $R(A_i) = R(A_j)$, then $A_i = A_j$

(3) if $R(A_i) > R(A_j)$, then $A_i > A_j$.

This method has some shortcomings. For example, let A and B be two triangular fuzzy numbers denoted by $A(3,4,5)$, $B(4,6,8)$, respectively (see Fig. 4.1).

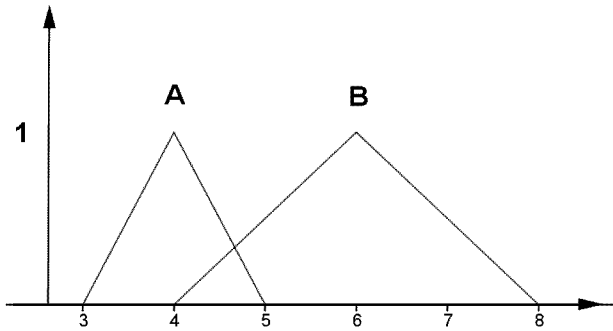


Fig.4.1

By Lemma 3.2, we have

$A(\bar{x}, \bar{y}) = (4, \frac{1}{3})$ and $B(\bar{x}, \bar{y}) = (6, \frac{1}{3})$. This implies

that $R(A) = 4.01$ and $R(B) = 6.01$. Thus, $R(A) < R(B)$ and $A < B$.

We can logically infer the ranking order of the images of these fuzzy numbers as $-A > -B$. However, by Cheng's method, the ranking order remains $-A < -B$.

4.2 Chu and Tsao's method [4].

Step 1. Find the centroid point (\bar{x}, \bar{y}) of fuzzy numbers.

Step 2. Calculate the area between the centroid point (\bar{x}, \bar{y}) and original point.

$$\text{i.e., } S(A) = \bar{x} \cdot \bar{y}$$

Step 3. Rank fuzzy numbers

(1) If $S(A_i) < S(A_j)$, then $A_i < A_j$

(2) If $S(A_i) = S(A_j)$, then $A_i = A_j$

(3) If $S(A_i) > S(A_j)$, then $A_i > A_j$.

This method can overcome Cheng's method, but still has some shortcomings. For example, let $A(3.7, 3.8, 4, 4)$ and $B(4, 5, 6)$ (see Fig.4.2).



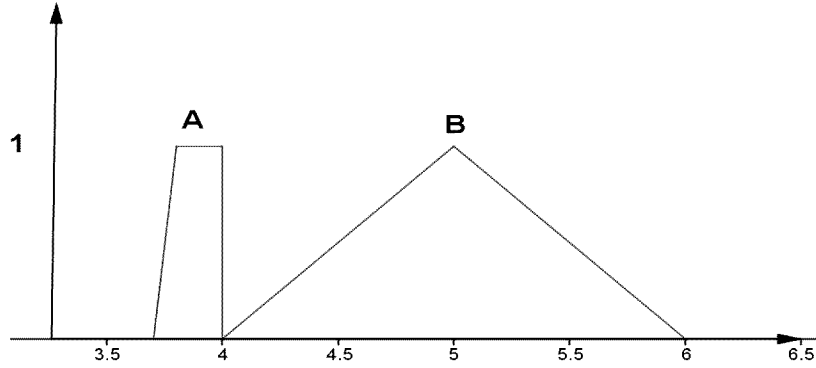


Fig.4.2

By Lemmas 2.2. and 2.3., we have

$$A(\bar{x}, \bar{y}) = (3.873, 0.467) \text{ and } B(\bar{x}, \bar{y}) = (5, \frac{1}{3}).$$

This implies that

$$S(A) = 1.81 \text{ and } S(B) = 1.67. \text{ Thus, } S(A) > S(B) \text{ and } A > B.$$

It is not reasonable. Chu and Tsao's problem is that the degree of importance of \bar{x} and \bar{y} is the same for ranking fuzzy numbers. However, they should have different degree of importance on a fuzzy number.

4.3 Xu and Wei's method [12].

Step 1. Find the centroid point (\bar{x}, \bar{y}) of fuzzy numbers.

Step 2. Calculate $A_\lambda = [(\mu_A^L)^{-1}(\lambda), (\mu_A^R)^{-1}(\lambda)]$

Step 3. Calculate

$$H(A) = \text{sign}(\bar{x}) \cdot (|\bar{x}| + \omega|\bar{y}| + \omega|\bar{x} \cdot \bar{y}|) \int_0^{\omega} (\mu_A^L)^{-1}(\lambda) d\lambda$$

where $\text{sign}(\cdot)$ is a symbolic function with $\text{sign}(x) = 1$ as $x \geq 0$ and $\text{sign}(x) = -1$ as $x < 0$.

Step 4. Rank fuzzy numbers, if $H(A) \leq H(B)$, then $A \leq B$.

In some situations, this method has some defects. For example, let $A(3, 5, 7)$ and $B(4, 5, 6)$ (see Fig. 4.3).

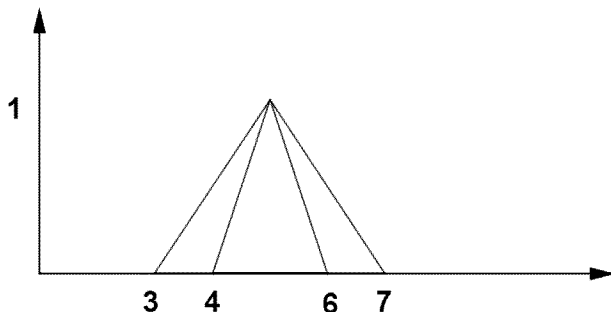


Fig.4.3

Clearly,

$$A(\bar{x}, \bar{y}) = (5, \frac{1}{3}), \quad H(A) = (5 + \frac{1}{3} + \frac{5}{3}) \cdot 4 = 28, \text{ and}$$

$$B(\bar{x}, \bar{y}) = (5, \frac{1}{3}), \quad H(B) = (5 + \frac{1}{3} + \frac{5}{3}) \cdot \frac{9}{2} = 31.5.$$

This implies that $H(A) < H(B)$ and $A < B$.

On the other hand, let $A(-2, 0, 2)$ and $B(-1, 0, 1)$ (see Fig. 4.4).

Clearly,

$$A(\bar{x}, \bar{y}) = (0, \frac{1}{3}), \quad H(A) = (0 + \frac{1}{3} + 0) \cdot 1 = 0.33, \text{ and}$$

$$B(\bar{x}, \bar{y}) = (0, \frac{1}{3}), \quad H(B) = (0 + \frac{1}{3} + 0) \cdot \frac{1}{2} = 0.167.$$

This implies that $H(A) > H(B)$ and $A > B$.

This result contradicts with the above.

4.4 Wang, Li and Liu's method [10].

Step 1. Calculate the left deviation degree S_i^L of fuzzy numbers A_i , $i = 1, 2, \dots, n$, where

$$S_i^L = \int_0^1 g_{A_i}^L(y) dy \text{ and } g_{A_i}^L \text{ is the inverse function of } \mu_{A_i}^L(x).$$

Step 2. Calculate the priority weight α_i

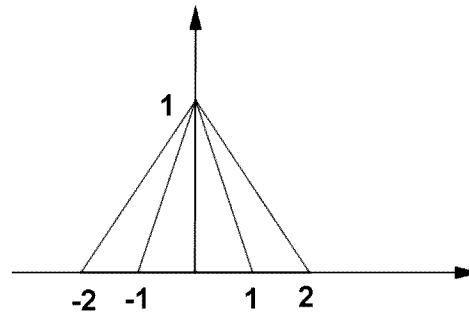


Fig.4.4



$$\alpha_i = \frac{S_i^L}{\sum_{i=1}^n S_i^L}$$

Step 3. Find the centroid point (\bar{x}, \bar{y}) of fuzzy numbers.

Step4. Get the dominance matrix

$$D = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ d_{21} & d_{22} & \dots & d_{2n} \\ \vdots & \vdots & \dots & \vdots \\ d_{n1} & d_{n2} & \dots & d_{nn} \end{bmatrix}, \text{ where } d_{ij} = \begin{cases} \frac{\bar{x}_i}{\bar{x}_j}, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

Step 5. Calculate the ranking index R_i

$$R_i = \lambda \cdot \sum_{j=1}^n d_{ij} \cdot d_{ij} + (1 - \lambda) \bar{y}_i, \text{ where}$$

$$\lambda \in [0.5, 1], \text{ generally, } \lambda = 0.5.$$

Step 6. Rank fuzzy numbers,

if $R_i < R_j$, then $A_i < A_j$

if $R_i = R_j$, then $A_i = A_j$.

The process is complicated and time-consuming. For example, let $A_1(3, 5, 7)$, $A_2(1, 2, 5)$, $A_3(3, 4, 7, 9)$ and $A_4(4, 9, 10)$ (see Fig.4.5).

Step 1. The left deviation degree

$$S_1^L = \int_0^1 g_{A_1}^L(y) dy = 4, \quad S_2^L = 1.5, \quad S_3^L = 3.5 \quad \text{and} \\ S_4^L = 6.5$$

Step 2. The priority weight

$$\alpha_i = \frac{S_i^L}{\sum_{i=1}^n S_i^L}, \left(\sum_{i=1}^4 S_i^L = 15.5 \right)$$

$$\alpha_1 = \frac{4}{15.5} = 0.26, \quad \alpha_2 = 0.10, \quad \alpha_3 = 0.23$$

and $\alpha_4 = 0.42$.

Step 3. The centroid point $(\bar{x}_1, \bar{y}_1) = (5, \frac{1}{3})$,

$(\bar{x}_2, \bar{y}_2) = (\frac{8}{3}, \frac{1}{3})$, $(\bar{x}_3, \bar{y}_3) = (5.78, \frac{4}{9})$

and $(\bar{x}_4, \bar{y}_4) = (\frac{23}{3}, \frac{1}{3})$.

Step 4. The dominance matrix

$$D = \begin{bmatrix} 1 & 1.88 & 0.87 & 0.65 \\ 0.53 & 1 & 0.46 & 0.35 \\ 1.16 & 2.17 & 1 & 0.75 \\ 1.53 & 2.88 & 1.33 & 1 \end{bmatrix}$$

Step5. The ranking index ($\lambda = 0.5$)

$$R_1 = \frac{1}{2} \times 0.26 \times (1 + 1.88 + 0.87 + 0.65) + \frac{1}{2} \times \frac{1}{3} = 0.739$$

$$R_2 = \frac{1}{2} \times 0.10 \times (0.53 + 1 + 0.46 + 0.35) + \frac{1}{2} \times \frac{1}{3} = 0.284$$

$$R_3 = \frac{1}{2} \times 0.23 \times (1.16 + 2.17 + 1 + 0.75) + \frac{1}{2} \times 0.44 = 0.804$$

$$R_4 = \frac{1}{2} \times 0.42 \times (1.53 + 2.88 + 1.33 + 1) + \frac{1}{2} \times \frac{1}{3} = 1.582$$

This implies that $R_4 > R_3 > R_1 > R_2$ and

$A_4 > A_3 > A_1 > A_2$.

V. REVISED RANKING METHOD

There are many researches about ranking fuzzy numbers based on centroid point. Almost each of them tried to use formulae to do this job. But because of the reasons they considered have different degree of importance. So, in some special conditions, the consequences don't make sense.

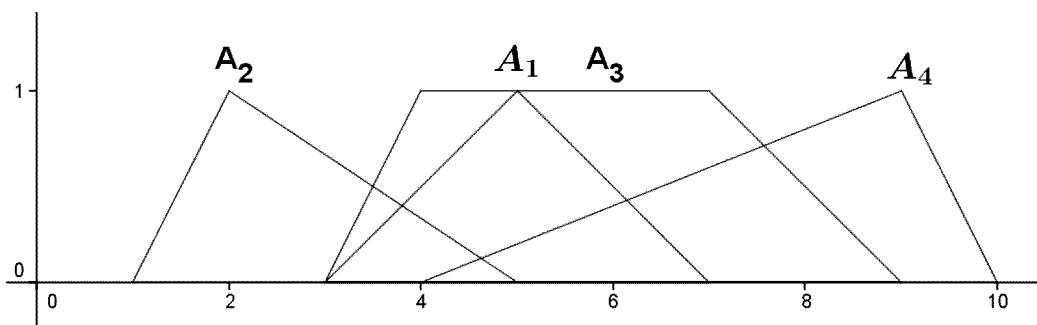


Fig.4.5



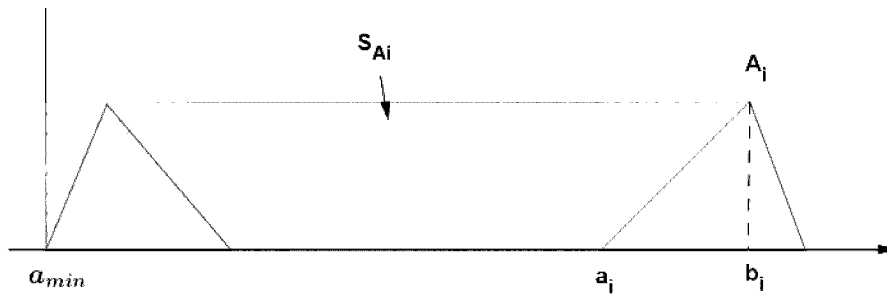


Fig.5.1

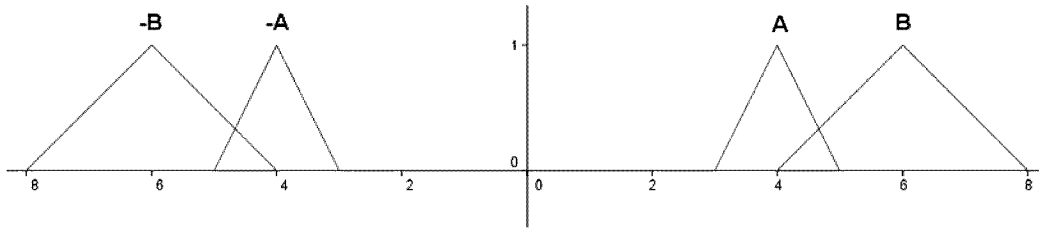


Fig.5.2

In this section, a new ranking method for trapezoidal fuzzy numbers is presented as follows.

Let $A_i = (a_i, b_i, c_i, d_i)$ be trapezoidal fuzzy

numbers, and \bar{x}_i is the horizontal ordinate of it's centroid point. By Lemma 3.3., we have

$$\bar{x}_i = \frac{1}{3} \left(a_i + b_i + c_i + d_i - \frac{c_i d_i - a_i b_i}{(c_i + d_i) - (a_i + b_i)} \right).$$

Let v_i be a point where the greatest membership function grade happened. Clearly, $v_i \in [b_i, c_i]$. Without loss of generality, let $v_i = \frac{b_i + c_i}{2}$.

Let S_i be the remainder area of A_i , i.e.,

$$S_i = (b_i - a_{\min}) \cdot 1 - \int_{a_i}^{b_i} \mu_{A_i}^L(x) dx, \text{ where}$$

$$a_{\min} = \min\{a_1, a_2, \dots, a_n\} \text{ (see Fig.5.1).}$$

We rank A_i and A_j based on their \bar{x}_i value. In the case that they are equal, we further compare their v_{A_i} value, and then S_{A_i} value to form their ranks.

Here is the algorithm of this method:

Step 1. Calculate the ranking index $C_p(A_i)$ for

$$A_i, \text{ where } C_p(A_i) = (\bar{x}_i, v_i, S_i).$$

Step 2. Rank fuzzy numbers

- (i) if $\bar{x}_{A_i} < \bar{x}_{A_j}$, then $A_i < A_j$.
- (ii) if $\bar{x}_{A_i} = \bar{x}_{A_j}$, then $A_i < A_j$ as $v_{A_i} < v_{A_j}$.

(iii) if $\bar{x}_{A_i} = \bar{x}_{A_j}$ and $v_{A_i} = v_{A_j}$, then $A_i < A_j$

as $S_{A_i} < S_{A_j}$.

Otherwise, $A_i = A_j$ as $S_{A_i} = S_{A_j}$.

Some examples that compared with previous methods are as follows.

Example 1 (compared with Cheng's method).

Let $A(3, 4, 5)$, $B(4, 6, 8)$, $-A(-5, -4, -3)$ and $-B(-8, -6, -4)$ (see Fig.5.2).

(1) Cheng's method: Compute $R(A) = \sqrt{(\bar{x})^2 + (v)^2}$.

we get $R(A) = 4.01 = R(-A)$, $R(B) = 6.01 = R(-B)$. It implies that $B = -B > A = -A$. It is unreasonable.

(2) The proposed method:

$A \cdot B$ can be represented by $A(3, 4, 4, 5)$, $B(4, 6, 6, 8)$, $-A(-5, -4, -4, -3)$, $-B(-8, -6, -6, -4)$. First, we calculate the ranking index and get

$$C_p(A) = \left(4, 4, \frac{23}{2} \right), \quad C_p(-A) = \left(-4, -4, \frac{5}{2} \right),$$

$$C_p(B) = \left(6, 6, 13 \right) \text{ and } C_p(-B) = \left(-6, -6, \frac{1}{2} \right)$$

By algorithm, it implies that $B > A > -A > -B$. It coincides with intuition.



Example 2 (compared with Chu and Tsao's method).

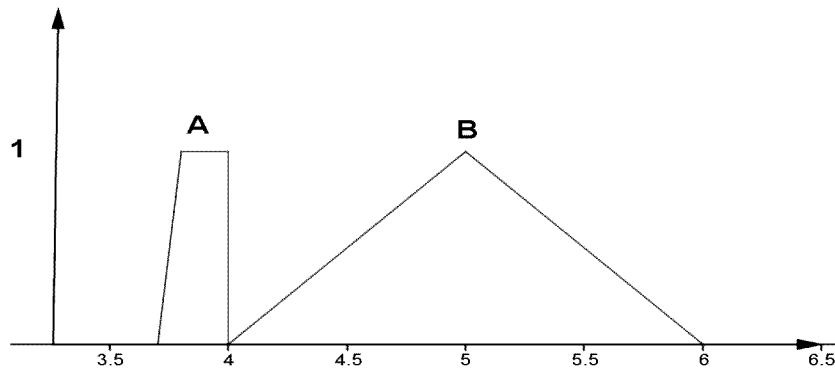


Fig.5.3

Let $A(3.7, 3.8, 4, 4)$ and $B(4, 5, 5, 6)$ (see Fig.5.3).

(1) Chu and Tsao's method: Compute $S(A) = \bar{x} \cdot \bar{y}$. We get $S(A)=1.81$, $S(B)=1.67$. It implies that $S(A) > S(B)$ and $A > B$. It is unreasonable.

(2) The proposed method:

We have the ranking index $C_p(A) = (3.87, 3.9, 0.05)$ and $C_p(B) = (5, 5, 0.8)$. It implies that $B > A$.

$$H(A) = \text{sign}(\bar{x}) \cdot \left(|\bar{x}| + \omega|\bar{y}| + \omega|\bar{x} \cdot \bar{y}| \right) \int_0^{\omega} \left(\mu_A^L \right)^{-1}(\lambda) d\lambda$$

$$H(A) = \left(0 + \frac{1}{3} + 0 \right) \cdot 1 = 0.33,$$

$$H(B) = \left(0 + \frac{1}{3} + 0 \right) \cdot \frac{1}{2} = 0.167,$$

$$H(C) = \left(5 + \frac{1}{3} + \frac{5}{3} \right) \cdot 4 = 28 \text{ and}$$

$$H(D) = \left(5 + \frac{1}{3} + \frac{5}{3} \right) \cdot \frac{9}{2} = 31.5.$$

It implies that $B < A < C < D$. It is unreasonable.

(2) The proposed method:

We have the ranking index $C_p(A) = (0, 0, 1)$,

$C_p(B) = (0, 0, 1.5)$, $C_p(C) = (5, 5, 6)$ and

$C_p(D) = (5, 5, 6.5)$. It implies that $A < B < C < D$.

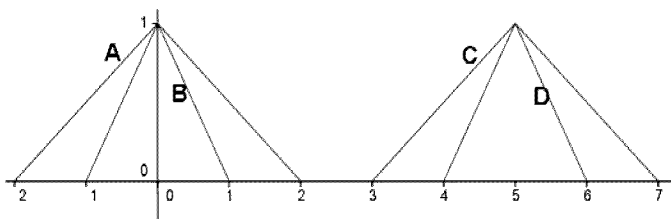


Fig.5.4

Let $A(-2, 0, 0, 2)$, $B(-1, 0, 0, 1)$, $C(3, 5, 5, 7)$ and $D(4, 5, 5, 6)$ (see Fig.5.4).

(1) Xu and Wei's method: compute $H(A)$. We get

Example 4 (compared with Wang, Li and Liu's method).

Let $A_1(3, 5, 5, 7)$, $A_2(1, 2, 2, 5)$, $A_3(3, 4, 7, 9)$ and $A_4(4, 9, 9, 10)$ (see Fig.5.5).

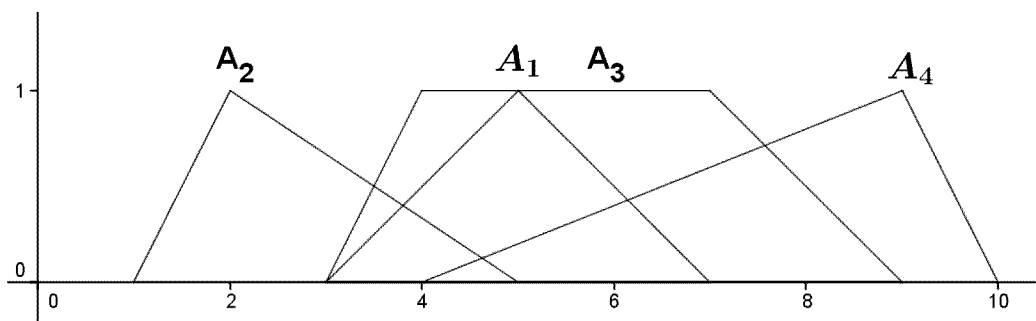


Fig.5.5



(1) Wang, Li and Liu's method: compute

$$R_i = \lambda \cdot \sum_{j=1}^n d_{ij} \cdot d_{ij} + (1-\lambda) \bar{y}_i.$$

We get $R_1 = 0.739$, $R_2 = 0.284$, $R_3 = 0.804$ and $R_4 = 1.582$.

It implies that $R_4 > R_3 > R_1 > R_2$ and

$$A_4 > A_3 > A_1 > A_2.$$

(2) The proposed method:

$$\text{We have the ranking index } C_p(A_1) = (5, 5, 3),$$

$$C_p(A_2) = (2.67, 2, 0.5), C_p(A_3) = (5.78, 5.5, 2.5)$$

$$\text{and } C_p(A_4) = (7.67, 9, 5.5). \text{ It implies}$$

$$\text{that } A_4 > A_3 > A_1 > A_2.$$

The outcome of ranking is the same with Wang, but our proposed method is easier and faster.

VI. CONCLUSION

Having reviewed the previous researches, we find that almost each of them try to use a formula to rank the fuzzy numbers. But they usually can not rank fuzzy numbers with human intuition consistently in all cases.

In this paper, a new method is proposed. It has two advantages in comparing it with the previous methods. The first one is that this new method is more consistent with our intuitions. The second is that the new method is easier and more efficient than others.

It uses three indices to differentiate between fuzzy numbers. The representation location of fuzzy number A is a top priority. The place that the largest membership function grade happened is the second priority. And then is S_A which means the remainder area is the last.

We can use these indices to rank fuzzy numbers validly. But this method remains to be further improved. Because it just can deal with normal fuzzy numbers at present.

When largest membership function grade is smaller than 1, it needs to consider the important degree of y-coordinate of the centroid that usually depend on the situation. Of course, it is more complicated but it can be the goal to striving for.

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BIOGRAPHIES

Ching-Ju Chiu is a student in Dept. of Applied Mathematics, National University of Tainan, and as a teacher in National Tainan Second Senior High School in Taiwan.

Bih-Sheue Shieh received the M. S. degree in mathematics from National Taiwan University, and the Ph. D. degree in applied mathematics from National Cheng Kung University in Taiwan. Currently, she is Professor in Dept. of Applied Mathematics, National University of Tainan in Taiwan.

